Another Approach to Estimation of Detection Threshold for Global Fringe Fitting.

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1 Intoduction

The purpose of this memo is to make more exact the analysis of the relationship between the number of (identical) antennas in the array and the threshold for point-source detection fulfiled by Cotton and Schwab [1]. Attention has been paid to the correlation between diferent "composite" baselines. This correlation has to decrease the threshold reduction factor deduced by Cotton and Schwab [1] and approach it to the A.E.E.E. Rogers' values [2].

2 Analysis of the threshold reduction factor

As demonstrated in [1] the detection threshold can be determined from the analysis of the phase fluctuation between a given antenna (a) and reference one (r). This phase difference can be estimated directly on single baseline ar or/and on multi baseline way $a.r$. The estimation of this phase difference can be described by:

$$\Phi_{ar} = \frac{1}{W}\left(\phi_{ar} + \frac{1}{2}\sum_{1}^{M}\phi_{a.r}\right) \tag{1}$$

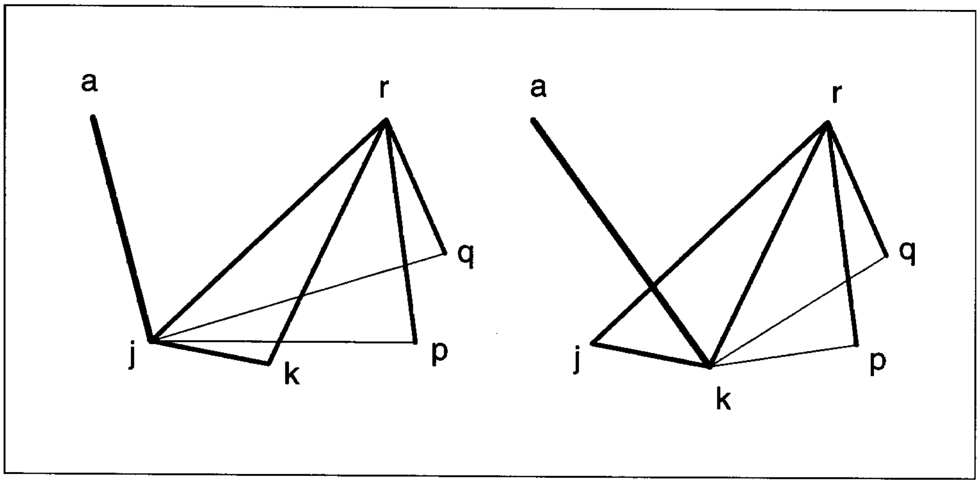
where W is total weight of the measuremenrs;
ϕ_{ar} is the phase estimation from the single baseline measurement;
$\phi_{a.r}$ is the phase estimation from a multi baseline measurement;
M is the number of multi baseline measurements; $M = n - 2$;
variances of multi baseline measurements (and therefore their weights) are assumed to be equal;

It is shown in [1] that the total weight $W = \frac{n}{2}$ if n is the number of antennas in the array. Assuming that the errors are uncorrelated we obtain the next equation for variance of measured phase difference Φ_{ar}:

$$D(\Phi_{ar}) = \left(\frac{2}{n}\right)^2 + D(\Phi_{a.r})(n-2)\left(\frac{1}{n}\right)^2 \tag{2}$$

When the multi baseline measurements are limited by two-baseline combinations, variance of a multi baseline measurement relative to a single baseline variance is $D(\Phi_{a.r}) = 2$. When we include three base line combination this variance is equal $1 + \frac{2}{n}$ [1]. Having substituted these values in (2) the expresions for variance of measured phase difference can be deduced:
$D(\Phi_{ar}) = \frac{2}{n}$, one- and two- baseline combinations;

$D(\Phi_{ar}) = \frac{1}{n} + \frac{4}{n^2} - \frac{4}{n^3}$, one-, two- and three baseline combinations;
These expresions were determined by [1] and they are true if all multi baseline measurements are uncorrelated. Indeed that is not true.



middle thickness' lines correspond to common baselines

Figure 1: Two realization of multi baseline measurements of phase between antenna 'a' and 'r'.

Two realizations of multi baseline measurements of phase between antenna 'a' and 'r' are shown in Fig. 1. These realizations correspond to the one-, two- and three baseline combinations described in [1]. It is clear that these two example are not uncorrelated because they have common baselines kj, jr, kr and pr, qr.. . At the same time it is clear that the degree of correlation is the same for all such pairs. Having noted to this type of correlation we can deduce the next expression for the variance of measured phase difference instead of (2):

$$D(\Phi_{ar}) = \left(\frac{2}{n}\right)^2 + D\left(\sum_{1}^{n-2} \Phi_{a.r}\right)\left(\frac{1}{n}\right)^2 = \left(\frac{2}{n}\right)^2 + D(\Phi_{a.r})(n-2)[1+(n-3)r]\left(\frac{1}{n}\right)^2 \qquad (3)$$

where $D(\Phi_{a.r})$ is the variance of phase through the multi baseline way ($aj.r$ or $ak.r$ at Fig. 1)
r is coefficient of correlation between measurements on the multi baseline way;

The phase estimation from the two multi baseline measurements shown in Fig. 1 can be expressed by the next equations:

$$\phi_{a.r1} = \phi_{aj} + \frac{2}{n-1}\left(\phi_{jr} + \frac{1}{2}\sum_{p=1}^{n-3} \phi_{jpr}\right) \qquad (4)$$

$$\phi_{a.r2} = \phi_{ak} + \frac{2}{n-1}\left(\phi_{kr} + \frac{1}{2}\sum_{p=1}^{n-3} \phi_{kpr}\right) \qquad (5)$$

We have here $n-1$ instead of n and $n-3$ instead of $n-2$ (comparatively with equation (1) because the antenna $'a'$ does not participate at the forming of multi baseline phase measurements and so we have one antenna less). Following analysis of [1] we can deduce the expresion for variance of multi baseline phase measurements:

$$D(\Phi_{a.r}) = 1 + \frac{2}{n-1} = \frac{n+1}{n-1} \qquad (6)$$

The statistical average of the product $\phi_{a.r1}\phi_{a.r2}$ can be determined from (4) and (5) after noting that baselines jk, kr and jr and $(n-3)$ of baselines pr are common:

$$\overline{\phi_{a.r1}\phi_{a.r2}} = \left(\frac{2}{n-1}\right)^2 \left[\frac{1}{2}\left(\overline{\phi_{jr}^2} + \overline{\phi_{kr}^2}\right) + \frac{1}{4}\left(\overline{\phi_{pr}^2}(n-3) - \overline{\phi_{jk}^2}\right)\right] = \frac{n}{(n-1)^2} \tag{7}$$

Having combined equations (6) and (7) we obtain the expression for coefficient of correlation r:

$$r = \frac{\overline{\phi_{a.r1}\phi_{a.r2}}}{D(\Phi_{a.r})} = \frac{n}{n^2-1} \tag{8}$$

Now we can substitude the expressions for $D(\Phi_{a.r})$ and r in (3) to obtain a final expression for the variance of measured phase difference:

$$D(\Phi_{ar}) = \left(\frac{2}{n}\right)^2 \left[1 + \frac{1}{4}(n-2)\left(1+\frac{2}{n-1}\right)\left(1+\frac{n(n-3)}{n^2-1}\right)\right] \approx \frac{2}{n}\left[1+\frac{(n-2)(n-3)}{2n(n-1)^2}\right] \tag{9}$$

The threshold reduction factor (TRF) can be found as square root of reciprocal of the variance of measured phase difference:

$$TRF = \frac{n(n-1)}{\sqrt{2n^3-3n^2-3n+6}} \approx \sqrt{\left(\frac{n}{2}\right)}\left[1-\frac{(n-2)(n-3)}{4n(n-1)^2}\right] \tag{10}$$

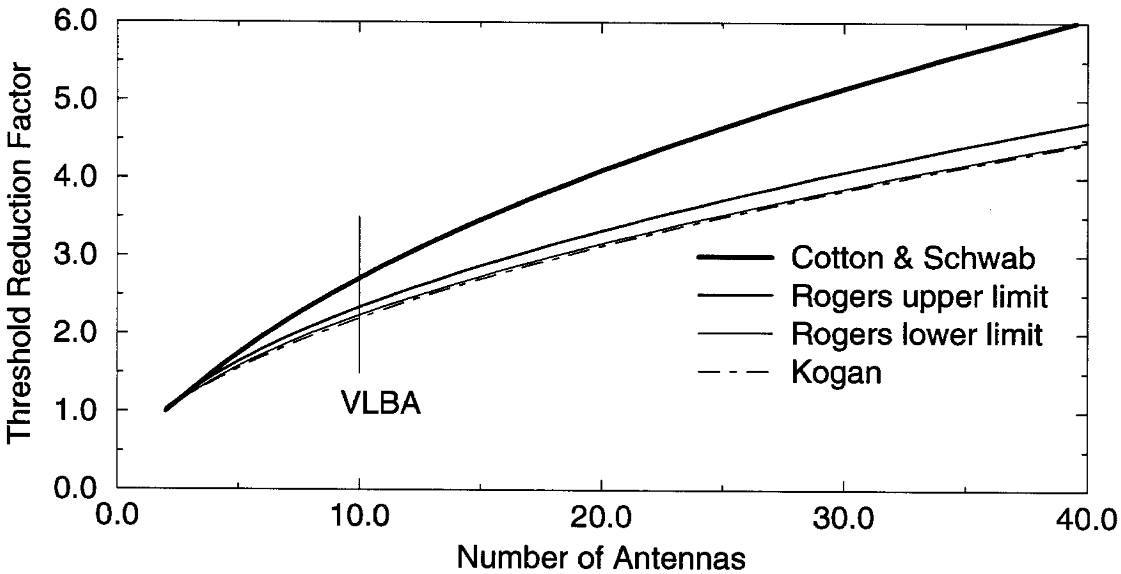


Figure 2: Threshold reduction factor according to different approaches

3 Discussion

Using only one and two baseline measurements provides $\sqrt{\frac{n}{2}}$ for TRF and $\frac{2}{n}$ for $D(\Phi_{ar})$ [1]. The equations (9) and (10) indicate that *as a result of large correlation between 'composite' baseline measurements its inclusion does not improve the variance of the estimated phase difference and TRF but on the contrary makes it worse a little bit.* The graph of the equation (10) as well as the analog graphs from [1] and [2] are shown in Fig. 2. The TRF graph passes very close to the A.E.E. Rogers lower limit curve [2]. In particular for the 10 elements VLBA the value 2.198 has been determined for the threshold reduction factor instead of 2.7 in [1] and (2.236-2.331) in [2]. The discrepancy between TRF and A.E.E. Rogers curve can be explained by differences between expected value for the noise peak used in [2] and noises variance used in [1] and here.

References

[1] W.D. Cotton and F.R. Scwab VLBA scientific memo 2, 1993

[2] Alan E.E. Rogers VLBA scientific memo 3, 1993