Effect of Digitizers Errors on the Cross and Auto Correlation Response of an FX Correlator

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1 Introduction

Both two and four level quantization schemes have been adopted in the design of the VLBA recording system. The conversion functions between the ideal correlation of the input analog signals and the measured correlation of the digitized signals are well known for all three possible combinations of the digitizers: TWO-TWO, TWO-FOUR, and FOUR-FOUR [1], [2], [3]. The parameters of the four level digitizer: the threshold v and the ratio of upper and intermediate levels n are chosen close to the optimal values $v=$ 0.9816, n=3.3359 [2], [3]. In real life it is difficult to have ideal digitizers with the exactly optimal values of these parameters. It is especially dificult to estimate precisely the threshold v measured in rms of input analog signal. That's why the effect of digitizer errors is especially strong in the case FOUR-FOUR and FOUR-TWO. The requirement on the acuracy of the digitizer parameters was formulated in [5]. This requirement is not satisfied sometimes. As a result the output of the VLBA correlator depends on the BBC and sample channel used. Compensation for this output deviation can be done using auto correlation measurements. In this memo we have found the relation between the deviation of the output cross and auto correlation spectra for the two combinations of the digitizers: FOUR-FOUR and FOUR-TWO.

2 Analysis of output of an FX correlator

The estimation of the relative error of the cross correlation for all three possible (in VLBA) combinations of digitizers is given by the equations [5]:

$$\frac{\Delta r_{44}}{r_{44}} = \frac{v}{1+(n-1)^{-1}e^{\frac{v^2}{2}}}(\epsilon_1+\epsilon_2) = 0.5798(\epsilon_1+\epsilon_2) \qquad (1)$$

$$\frac{\Delta r_{24}}{r_{24}} = \frac{v}{1+(n-1)^{-1}e^{\frac{v^2}{2}}}\epsilon = 0.5798\epsilon \qquad (2)$$

$$\frac{\Delta r_{22}}{r_{22}} = \frac{1}{2}\left(c_1^2+c_2^2\right) \qquad (3)$$

where $n = 3.3359$ is the optimal ratio of the levels in four levels digitizers.
$v = 0.9816$ is the optimal value of the threshold in four levels digitizers.
ϵ_1, ϵ_2 are input power level (gain) errors.
c_1, c_2 are errors in the determination of zero line of the two level digitizers.

Sometimes the quality of a digitizer is estimated by a deviation of a probability of a digitized signal being on a given level from the nominal value of the probability (state count). The relation between threshold's error and the probability deviation is given in Appendix 3. A zero line can be determined more precisely than the rms of an input analog signal (hence the parameter v). Furthermore errors in the determination of zero lines enter in the square into equation (3). Therefore the effect of digitizer errors is not so strong in the case of two level digitizers on both antennas of an interferometer. That's why we exclude the TWO-TWO case from our analysis and concentrate our attention only on the FOUR-FOUR and FOUR-TWO cases.

In the case of low correlation (the most typical case in radio astronomy) the relative error of correlation equals the relative error of the estimated amplitude in the spectrum measured by an FX correlator. So the equations (1), (2) work for estimation of the specrum as well.

$$\frac{\Delta A_{44}}{A_{44}} = 0.5798(\epsilon_i + \epsilon_j) \tag{4}$$

$$\frac{\Delta A_{24}}{A_{24}} = 0.5798\epsilon \tag{5}$$

Unknown errors of digitizers ϵ_i, and ϵ_j can be evaluated from auto correlation measurements.

Continuum Sources. Flat Spectrum.

Let's suppose the amplitude-frequency pattern of the spectrum of correlated signals has an ideal rectangular form. Then the amplitude of the measured auto correlation spectrum is completely determined by the value of the auto correlation at zero delay, because values of the auto correlation function in other sampling delays are equal zero. Therefore the relative error of the auto correlation spectrum equals the relative error of the square of the digital signal and can be calculated using equation (19) of the Appendix 1.

$$\frac{\Delta A_i}{A_i} = 0.5798 \cdot 2\epsilon_i \tag{6}$$

It is remarkable that the coefficients in the formulas for auto and cross correlation (0.5798) are identical for optimal values of n and v. Having combined formulas (4), (5) with (6) we obtain the final expression relating errors in measurement of auto and cross correlation spectra.

$$\frac{\Delta A_{44}}{A_{44}} = 0.5\left(\frac{\Delta A_i}{A_i} + \frac{\Delta A_j}{A_j}\right) \tag{7}$$

$$\frac{\Delta A_{24}}{A_{24}} = 0.5\frac{\Delta A_i}{A_i} \tag{8}$$

Spectral Line Sources. Arbitrary Spectrum.

In this case, the auto correlation coeficients of the input analog signal can have any value in the sampling delays. Estimation of the relative error of auto correlation of a digitized signal for arbitrary correlation can be done using the equtions (26) of Appendix 2 and values of auto correlation of a digitized signal $r_{44}(\rho)$ which are given in table form in [3]:

$$\begin{aligned} \frac{\Delta r_{44}(\rho)}{r_{44}(\rho)} &= (r_{44}(\rho))^{-1}\left(-\frac{2(n-1)}{\sqrt{2\pi}}\exp(-\frac{v^2}{2})\right)\cdot \\ &\left[erf\left(\frac{\rho v}{\sqrt{1-\rho^2}\sqrt{2}}\right) + (n-1)\frac{1}{2}\left(erf\left(\frac{\rho v+v}{\sqrt{1-\rho^2}\sqrt{2}}\right) + erf\left(\frac{\rho v-v}{\sqrt{1-\rho^2}\sqrt{2}}\right)\right)\right]\cdot 2\epsilon \\ &= C\cdot 2\epsilon \end{aligned} \tag{9}$$

We have calculated the coefficient C in (9). The results of the calculation are given in table form in table 1 and in plot form in Fig.1. We see from table 1 and Fig.1 that the relative error of the auto correlation

Table 1: The values of coefficien C at equation $\frac{\Delta r_{44}(\rho)}{r_{44}(\rho)} = C \cdot 2\epsilon$

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C	0.5798	0.5804	0.5824	0.5858	0.5905	0.5964	0.6036	0.6119	0.6209	0.6265	0.5798

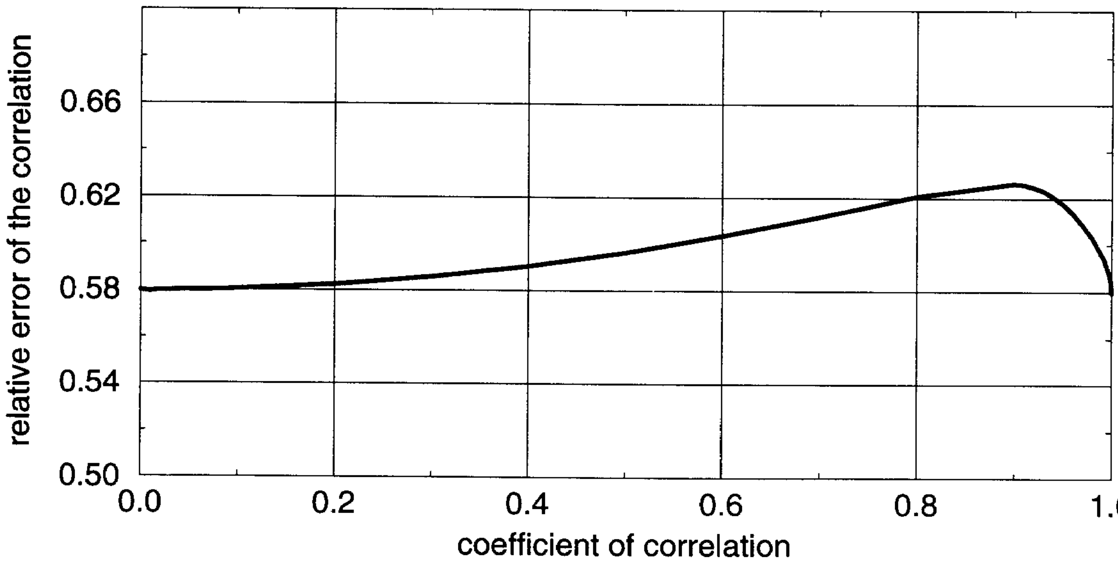

Figure 1: The relative error of the correlation due to errors of the digitizer via the correlation itself

is identical to that of the cross correlation with accuracy better 0.1% if the relevant analog correlation is less than 0.1 or is equal 1. The fact that $\frac{\Delta r_{44}(\rho)}{r_{44}(\rho)} \neq const$ in the whole range of ρ complicates the correction for the digitizing of the signals. We can't apply the same conversion function for all values of ρ but have to use a different correction for different ρ in accordance with table 1 or Fig.1. This correction is needed only in the case of extremely powerful spectral line sources and big digitizer errors.

So, we have proved that the relative error of the auto correlation of a digitized signal is determined by the same formulas for the case of full correlation of the input analog signal (zero delay) as well as for small correlation. If we exclude the rare case of spectral line sources with the line powers comparable to noise power, the auto correlation is less than 0.1 at the discrete values of the delays $\pm\frac{n}{2\Delta f}$. That is why we infer that the relative error of the auto correlation spectrum is equal to relative error of the auto correlation and can be determined by the linear equation:

$$\frac{\Delta A_i(f)}{A_i(f)} = 0.5798 \cdot 2\epsilon_i \qquad (10)$$

This equation coincides with relevant equation for continuum spectrum. That's why the equations relating the errors in the cross and auto correlation spectra (7), (8) work here as well. But the problem is how to determine the deviation in the auto correlation spectrum, because we do not know the correct spectrum. The problem can be circumvented if we calculate the digitizer's error ϵ_i, using the deviation of mean level of the measured auto correlation spectrum. Indeed the mean level is proportional to the mean of square of the digitized signal and has to be the same for any input gaussian random process independent on its spectrum. This level is known in advance. It is equal 1 after applying the digital correction corresponding to an "ideal" digitizer. Therefore finally we can deduce the next expression for estimating relative error of cross correlation spectrum:

$$\frac{\Delta A_{44}(f)}{A_{44}(f)} = 0.5\left[\left(\frac{\Delta A_i}{A_i}\right)_m + \left(\frac{\Delta A_j}{A_j}\right)_m\right] \tag{11}$$

$$\frac{\Delta A_{24}(f)}{A_{24}(f)} = 0.5\left(\frac{\Delta A_i}{A_i}\right)_m \tag{12}$$

where $\left(\frac{\Delta A_i}{A_i}\right)_m$ is the relative error of the mean of the auto correlation spectrum.

The auto correlation spectrum of the spectral line source itself $A_s(f)$ is determined usually by subtracting the ON and OFF source specrum:

$$A_s(f) = \frac{A_{on}(f) - A_{off}(f)}{A_{off}(f)} \tag{13}$$

Because the deviation of the auto correlation spectrum is determined by the same linear function of digitizer error (10), the spectrum of the source itself $A_s(f)$ does not depend on the digitizer error at all.

3 Conclusion

The correction of an auto correlation of a digitized signal for the errors of the digitizer has been calculated as a function of the correlation for a FOUR level digitizer. This function is linear with the same slope for small correlation (less ~ 0.1) and for full correlation. The correction of a measured cross correlation spectrum due to the errors of the digitizers can be done, using the measurement of auto correlation spectra. It has been shown that an error of a digitizer's threshold can be estimated measuring the deviation of the mean of the auto correlation spectrum. The nominal value of the mean does not depend on the spectrum and equals 1 after applying digital correction corresponding to an "ideal" digitizer.

APPENDIX 1
EFFECT OF DIGITIZER ERRORS ON THE SQUARE OF A DIGITIZED SIGNAL FOUR LEVEL DIGITIZER.

The operation of four level quantizer of the signal is described by a step function $q(\xi)$:

$$\hat{\xi} = q(\xi) = \begin{cases} +n & \text{if} \quad \xi \geq a \cdot \sigma \\ +1 & \text{if} \quad 0 \leq \xi < a \cdot \sigma \\ -1 & \text{if} \quad b \cdot \sigma \leq \xi < 0 \\ -n & \text{if} \quad \xi < b \cdot \sigma \end{cases} \tag{14}$$

where σ is the variance of the input analog random process $\xi(t)$,
a, b are the upper and low thresholds of the digitizer measured at number of σ,
n is the ratio of the levels.

In an ideal case $a = -b = v = 0.9816$, n =3.336. In fact the thresholds a, and $-b$ are different:

$$\begin{aligned} a = &\quad v + \alpha = &\quad v + (\epsilon + \delta) \\ b = &\quad -v - \beta = &\quad -v - (\epsilon - \delta) \end{aligned} \tag{15}$$

where δ is DC offset of the digitizer
ϵ is input power level's (sampler gain) variation.

It is clear from equation (14) that mathematical expectation of the square of digitizer's output (auto correlation in zero delay) equals:

$$\overline{\hat{\xi}^2} = n^2 \cdot \int_a^{\infty} + 1 \cdot \int_0^a + 1 \cdot \int_b^0 + n^2 \cdot \int_{-\infty}^b \tag{16}$$

where $\int_c^d = \frac{1}{\sqrt{2\pi}} \int_c^d e^{\frac{-t^2}{2}} dt$
We consider that the input analog signal is represented by gaussian noise with zero mean and unity rms. The deviation of $\overline{\hat{\xi}^2}$ can be calculated having taken the full differential of expression (16) using the deviation a near v and deviation b near $-v$.

$$\Delta\overline{\hat{\xi}^2} = \frac{\partial(\overline{\hat{\xi}^2})}{\partial a}\Delta a + \frac{\partial(\overline{\hat{\xi}^2})}{\partial b}\Delta b \tag{17}$$

Having evaluated the derivatives in (17) we obtain the next expression for the deviation of auto correlation output

$$\begin{aligned} \Delta\overline{\hat{\xi}^2} &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}) \left[n^2(-\Delta a) + \Delta a - \Delta b + n^2 \Delta b\right] \\ &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})(n^2 - 1)(-\Delta a + \Delta b) \\ &= -\frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})(n^2 - 1)2\epsilon \end{aligned} \tag{18}$$

Having substituted the optimal values $v = 0.9816$, $n = 3.33587$ and normalized to maximum auto correlation output $\overline{\hat{\xi}^2} = 4.3048$ [3], we obtain the final expression for the relative deviation of auto correlation output

$$\frac{\Delta\overline{\hat{\xi}^2}}{\overline{\hat{\xi}^2}} = 0.5798 \cdot 2\epsilon \tag{19}$$

APPENDIX 2
EFFECT OF DIGITIZERS ERRORS ON THE AUTO CORRELATION. ARBITRARY VALUE OF THE CORRELATION. FOUR LEVEL DIGITIZER.

The expression for the correlation of the four levels digitized signals $r(\rho)$ is represented by the next equation [2]:

$$\begin{aligned} r(\rho) = \quad & (n-1)^2[L(a,a,\rho)+2L(a,b,\rho)+L(b,b,\rho)+1] \\ & +4(n-1)[L(a,0,\rho)+L(b,0,\rho)] \\ & -2n(n-1)[Q(a)+Q(b)]+\tfrac{2}{\pi}\arcsin\rho \end{aligned} \tag{20}$$

where $L(h,k,\rho)=\int_h^\infty\int_k^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)}\,dx\,dy$

$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-t^2}{2}}\,dt$

a, b thresholds of the digitizer detemined in Appendix 1 (equation (15))

The expression for the error of $r(\rho)$ can be written in the form of full differential relatively of the digitizer's threshold errors Δa, Δb:

$$\Delta r = \frac{\partial r}{\partial a}\Delta a + \frac{\partial r}{\partial b}\Delta b \tag{21}$$

Let's start calculating of $\frac{\partial r}{\partial a}$ from (20)

$$\begin{aligned} \frac{\partial r}{\partial a} &= (n-1)^2\left[\frac{\partial L(a,a,\rho)}{\partial a}+2\frac{\partial L(a,b,\rho)}{\partial a}\right]+4(n-1)\frac{\partial L(a,0,\rho)}{\partial a}-2n(n-1)\frac{\partial Q(a)}{\partial a} \\ &= (n-1)^2\left[-\int_a^\infty g(a,y,\rho)\,dy-\int_a^\infty g(x,a,\rho)\,dx-2\int_b^\infty g(x,a,\rho)\,dx\right] \\ & -4(n-1)\int_0^\infty g(x,a,\rho)\,dx+\frac{2n(n-1)}{\sqrt{(2\pi)}}\exp(-\tfrac{a^2}{2}) \end{aligned} \tag{22}$$

Taking into account a symmetry of the function $g(x,y,\rho)$ relatively x,y , we can simplify equation (22) having carryed out a change of integration variable $z=\frac{x-\rho a}{\sqrt{1-\rho^2}}$:

$$\begin{aligned} \frac{\partial r}{\partial a} &= -\frac{2(n-1)}{\sqrt{2\pi}}\exp(-\tfrac{a^2}{2})\cdot \\ & \left[erf\left(\frac{\rho a}{\sqrt{1-\rho^2}\sqrt{2}}\right)+(n-1)\tfrac{1}{2}\left(erf\left(\frac{\rho a+a}{\sqrt{1-\rho^2}\sqrt{2}}\right)+erf\left(\frac{\rho a+b}{\sqrt{1-\rho^2}\sqrt{2}}\right)\right)\right] \end{aligned} \tag{23}$$

Having substituted nominal values $a=v$ and $b=-v$ we obtain final expression for $\frac{\partial r}{\partial a}$:

$$\begin{aligned} \frac{\partial r}{\partial a} &= -\frac{2(n-1)}{\sqrt{2\pi}}\exp(-\tfrac{v^2}{2})\cdot \\ & \left[erf\left(\frac{\rho v}{\sqrt{1-\rho^2}\sqrt{2}}\right)+(n-1)\tfrac{1}{2}\left(erf\left(\frac{\rho v+v}{\sqrt{1-\rho^2}\sqrt{2}}\right)+erf\left(\frac{\rho v-v}{\sqrt{1-\rho^2}\sqrt{2}}\right)\right)\right] \end{aligned} \tag{24}$$

Providing the same analisys for $\frac{\partial r}{\partial b}$, we can prove that:

$$\frac{\partial r}{\partial b} = -\frac{\partial r}{\partial a} \tag{25}$$

Having combined equations (21), (24) and (25) we can deduce the expression for the deviation of correlation of digitized signal as a function of correlation of relevant analog signals:

$$\begin{aligned} \Delta r &= \frac{\partial r}{\partial a}(\Delta a-\Delta b) \\ &= -\frac{2(n-1)}{\sqrt{2\pi}}\exp(-\tfrac{v^2}{2})\cdot \\ & \left[erf\left(\frac{\rho v}{\sqrt{1-\rho^2}\sqrt{2}}\right)+(n-1)\tfrac{1}{2}\left(erf\left(\frac{\rho v+v}{\sqrt{1-\rho^2}\sqrt{2}}\right)+erf\left(\frac{\rho v-v}{\sqrt{1-\rho^2}\sqrt{2}}\right)\right)\right]\cdot 2\epsilon \end{aligned} \tag{26}$$

APPENDIX 3
RELATION BETWEEN THRESHOLD AND STATE COUNTS.

The operation of four level quantizer of the signal is described by a step function $q(\xi)$:

$$\hat{\xi} = q(\xi) = \begin{cases} +n & \text{if} \quad \xi \geq a \cdot \sigma, & Level \ I \\ +1 & \text{if} \quad 0 \leq \xi < a \cdot \sigma, & Level \ II \\ -1 & \text{if} \quad b \cdot \sigma \leq \xi < 0, & Level \ III \\ -n & \text{if} \quad \xi < b \cdot \sigma, & Level \ IV \end{cases} \tag{27}$$

where σ is the variance of the input analog random process $\xi(t)$,
a, b are the upper and low thresholds of the digitizer measured at number of σ,
n is the ratio of the levels.

In an ideal case $a = -b = v = 0.9816$, n =3.336. In fact the thresholds a, and $-b$ are different:

$$\begin{aligned} a = \quad & v + \alpha = \quad v + (\epsilon + \delta) \\ -b = \quad & v + \beta = \quad v + (\epsilon - \delta) \end{aligned} \tag{28}$$

where α, β are errors of upper and lower thresholds
δ is DC offset of the digitizer
ϵ is input power level's (sampler gain) variation.

The thresholds are measured in number of σ. That is why we can consider that the probability distribution of an input analog signal is a normal distribution with zero mean and unity rms. So the probabilities of signal's being on the levels I, I, III, IV (relative counts) are determined by the next equations:

$$\begin{aligned} P(I) &= \int_a^\infty &= \tfrac{1}{2} - P(II) \\ P(II) &= \int_0^a &= \tfrac{1}{2} \cdot erf(\tfrac{a}{\sqrt{2}}) \\ P(III) &= \int_0^b &= \tfrac{1}{2} \cdot erf(\tfrac{b}{\sqrt{2}}) \\ P(IV) &= \int_b^\infty &= \tfrac{1}{2} - P(III) \end{aligned} \tag{29}$$

where $\int_c^d = \frac{1}{\sqrt{2\pi}} \int_c^d e^{\frac{-t^2}{2}} dt$

Deducing the equation (29) we suppose that zero line of the input random process has been determined ideally. Now we consider the errors of thresholds to be small in comparison with the thresholds' values. Then the integrals in (29) can be simplified:

$$\begin{aligned} P(I) &= \tfrac{1}{2} - P_0(II) - \tfrac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \cdot \alpha &&= P_0(I) + \Delta P(I) \\ P(II) &= \int_0^v + \tfrac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \cdot \alpha &&= P_0(II) + \Delta P(II) \\ P(III) &= \int_0^v + \tfrac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \cdot \beta &&= P_0(III) + \Delta P(III) \\ P(IV) &= \tfrac{1}{2} - P_0(III) - \tfrac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \cdot \beta &&= P_0(IV) + \Delta P(IV) \end{aligned} \tag{30}$$

Substituting the nominal value of the threshold $v = 0.9816$ in (30), we can obtain the following expressions for the nominal values of the probabilities and their deviations depending on the thresholds' errors.

$$\begin{aligned} P(I) &= 0.1631 \\ P(II) &= 0.3369 \\ P(III) &= 0.3369 \\ P(IV) &= 0.1631 \end{aligned} \tag{31}$$

$$\begin{aligned} \Delta P(I) &= -0.246 \Delta\alpha \\ \Delta P(II) &= 0.246 \Delta\alpha \\ \Delta P(III) &= 0.246 \Delta\beta \\ \Delta P(IV) &= -0.246 \Delta\beta \end{aligned} \tag{32}$$

So for example, a threshold's error 20% corresponds to a deviation in the probability of 0.05.

References

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