Fundamentals of Radio Interferometry

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Fourteenth NRAO Synthesis Imaging Summer School

Socorro, NM

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Associated Universities, Inc.

Topics

- Why Interferometry?
- The Single Dish … as an interferometer
- The Basic Interferometer
  - Response to a Point Source
  - Response to an Extended Source
  - The Complex Correlator
  - The Visibility and its relation to the Intensity
  - Picturing the Visibility

Why Interferometry?

- Because of Diffraction: For an aperture of diameter D, and at wavelength l, the image resolution is

$$\theta_{rad} \approx \lambda / D$$

- In 'practical' units:

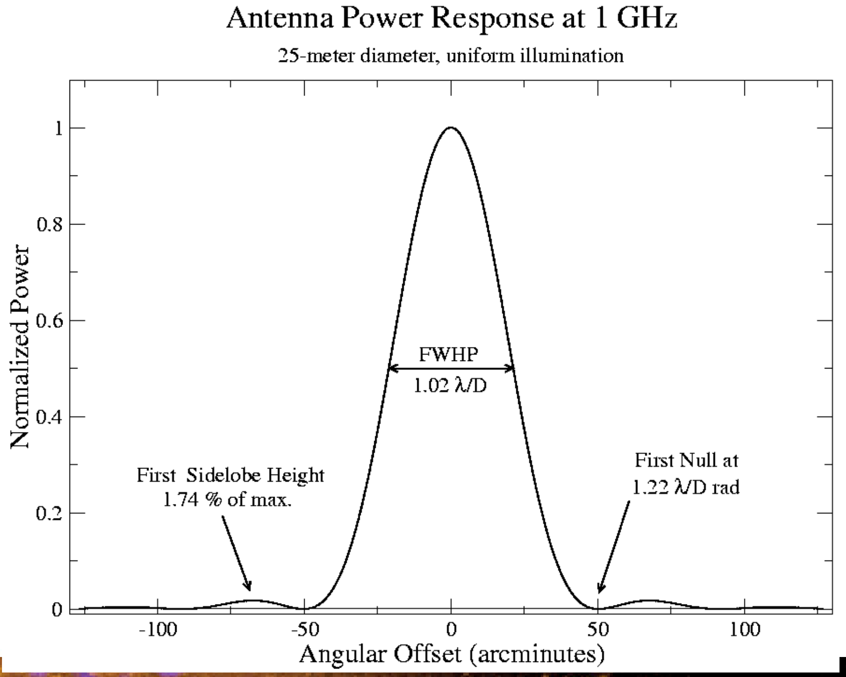
$$\theta_{\mathrm{arcsec}} \approx 2\,\lambda_{\mathrm{cm}} / D_{\mathrm{km}}$$

- To obtain 1 arcsecond resolution at a wavelength of 21 cm, we require an aperture of ~42 km!
- The (currently) largest single, fully-steerable apertures are the 100 meter antennas near Bonn, and at Green Bank.
- So we must develop a method of synthesizing an equivalent aperture.
- The methodology of synthesizing a continuous aperture through summations of separated pairs of antennas is called 'aperture synthesis'.

The Single Dish – as in Interferometer

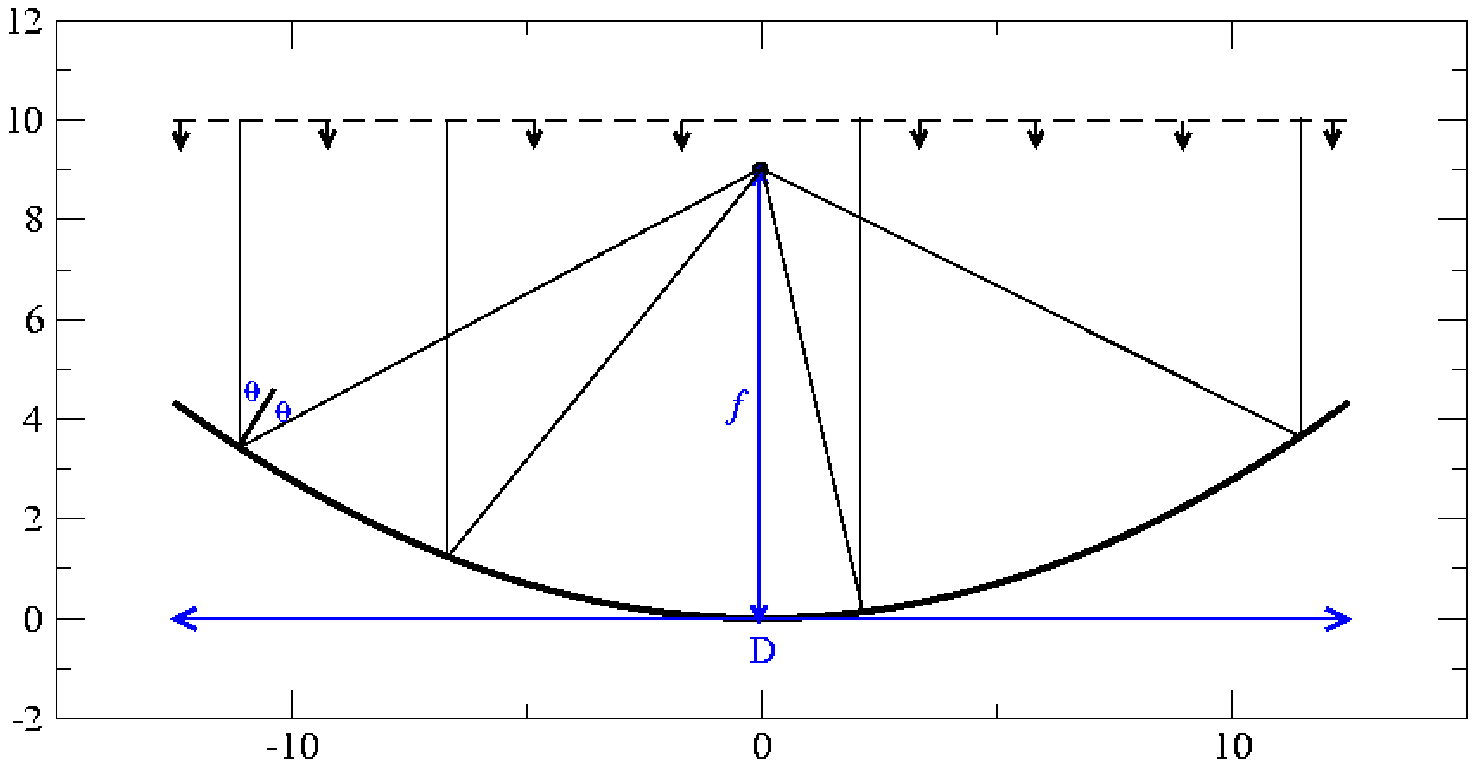
- A parabolic reflector has a power response (vs angle) roughly as shown below.
- The formation of this response follows the same laws of physics as an interferometer.
- A basic understanding of the origin of the focal response will aid in understanding how an interferometer works.

Illustrated here is the approximate power response of a 25-meter antenna, at ν = 1 GHz.



The Parabolic Reflector

- Key Point: Distance from incoming phase front to focal point is the same for all rays.
- The E-fields will thus all be in phase at the focus – the place for the receiver.

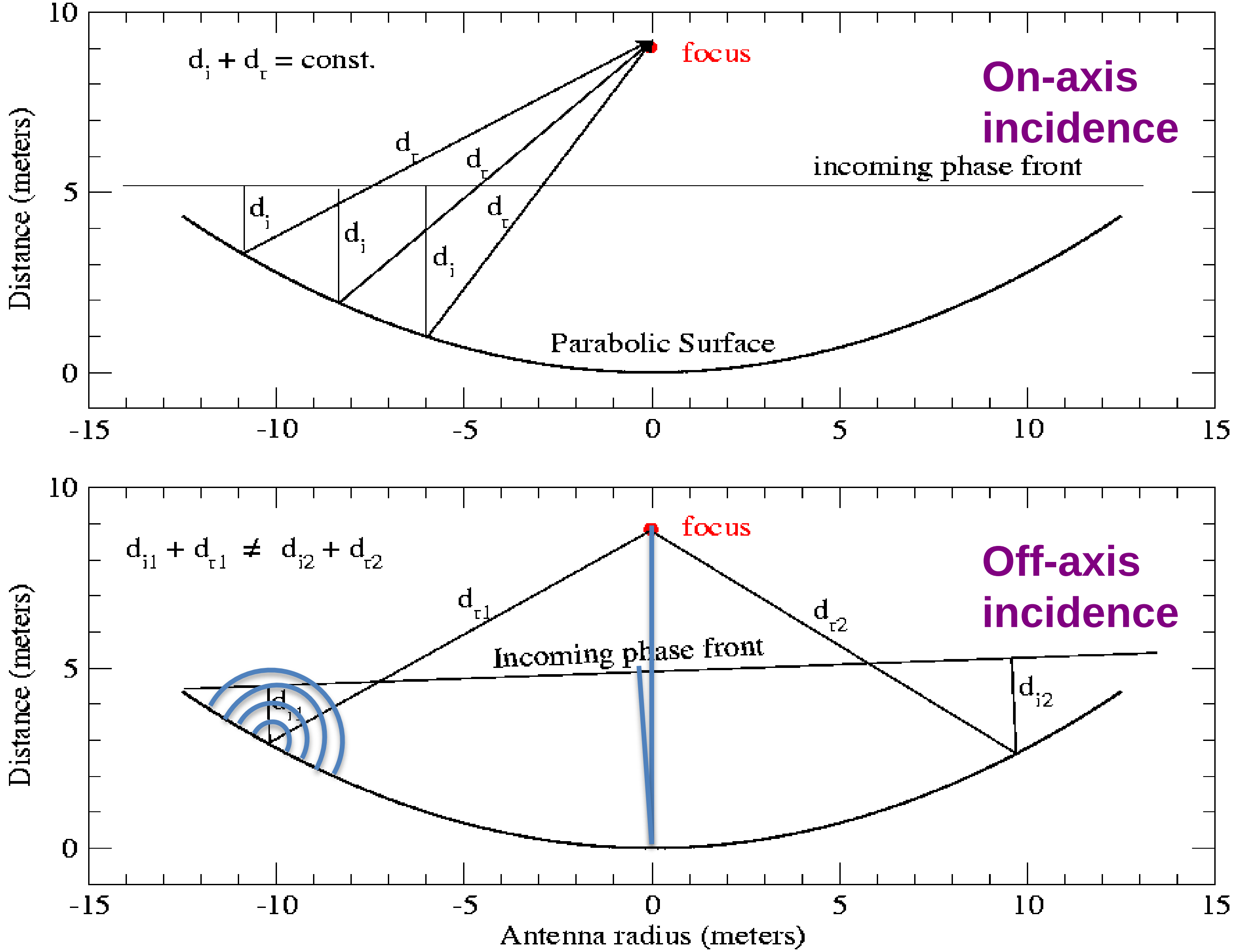


Beam Pattern Origin
(1-Dimensional Example)

- An antenna's response is a result of coherent vector summation of the electric field at the focus.
- First null will occur at the angle where one extra wavelength of path is added across the full width of the aperture:

$\theta \sim \lambda/D$

(Why?)



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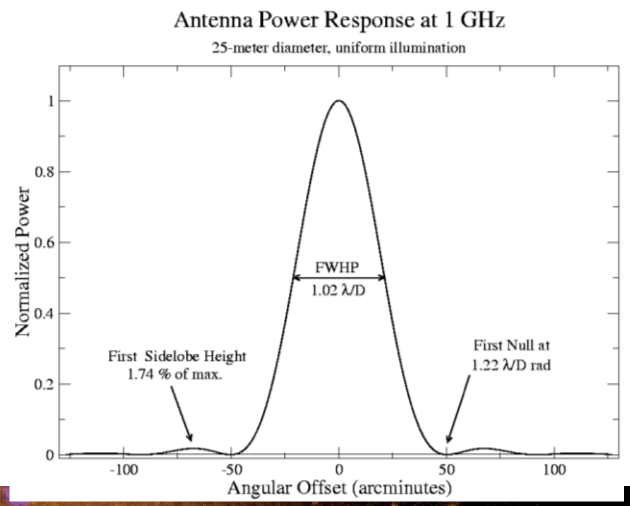
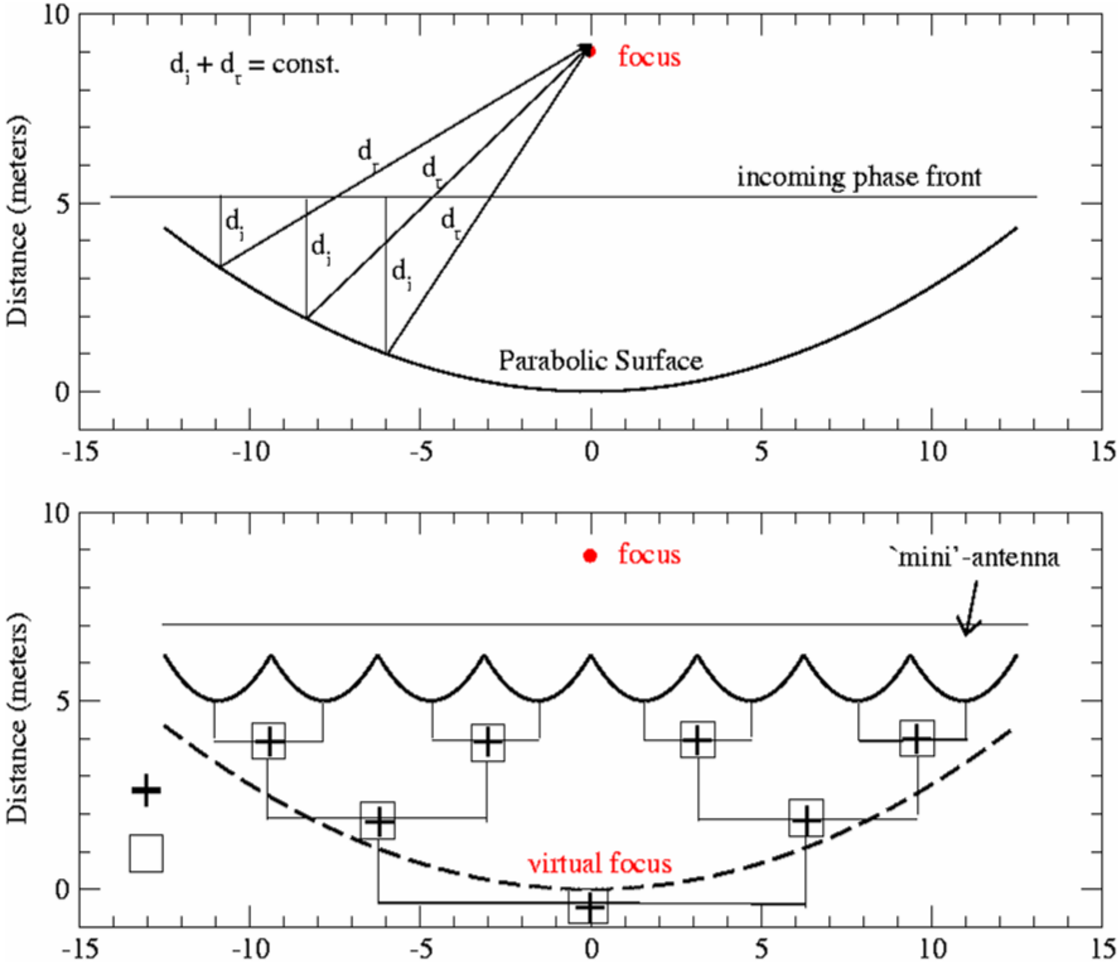
Specifics: First Null, and First Sidelobe

- When the phase differential across the aperture is 1, 2, 3, … wavelengths, we get a null in the total received power.
  - The nulls appear at (approximately): $\theta = \lambda/D, 2\lambda/D, 3\lambda/D, \ldots$ radians.
- When the phase differential across the full aperture is ~1.5, 2.5, 3.5, … wavelengths, we get a maximum in total received power.
  - These are the 'sidelobes' of the antenna response.
  - But, each successive maximum is weaker than the last.
  - These maxima appear at (approximately): $\theta = 3\lambda/2D, 5\lambda/2D, 7\lambda/2D, \ldots$ radians.



Interferometry – Basic Concept

- We don't need a single parabolic structure.
- We can consider a series of small antennas, whose individual signals are summed in a network.
- This is the basic concept of interferometry.
- Aperture Synthesis is an extension of this concept.



Quasi-Monochromatic Radiation

- Analysis is simplest if the fields are perfectly monochromatic.
- This is not possible – a perfectly monochromatic electric field would both have no power ($\Delta\nu = 0$), and would last forever.
- So we consider instead 'quasi-monochromatic' radiation, where the bandwidth $\delta\nu$ is very small.
- For a time $dt \sim 1/d\nu$, the electric fields will be sinusoidal.
- Consider then the electric fields from a small sold angle $d\Omega$ about some direction **s**, within some small bandwidth $d\nu$, at frequency ν.
- We can write the temporal dependence of this field as:

$$E_\upsilon(t) = A\cos(2\pi\upsilon t + \phi)$$

- The amplitude and phase remains unchanged to a time duration of order $dt \sim 1/d\nu$, after which new values of **A** and ϕ are needed.

Simplifying Assumptions

- We now consider the most basic interferometer, and seek a relation between the characteristics of the product of the voltages from two separated antennas and the distribution of the brightness of the originating source emission.
- To establish the basic relations, the following simplifications are introduced:
  - Fixed in space – no rotation or motion
  - Quasi-monochromatic (signals are sinusoidal)
  - No frequency conversions (an 'RF interferometer')
  - Single polarization
  - No propagation distortions (no ionosphere, atmosphere …)
  - Idealized electronics (perfectly linear, no amplitude or phase perturbations, perfectly identical for both elements, no added noise, …)

Symbols Used, and their Meanings

- We consider:
  - two identical sensors, separated by vector distance **b**
  - receiving signals from vector direction **s**
  - at frequency ν (angular frequency $\omega = 2\pi\nu$)
- From these, the key quantity

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$

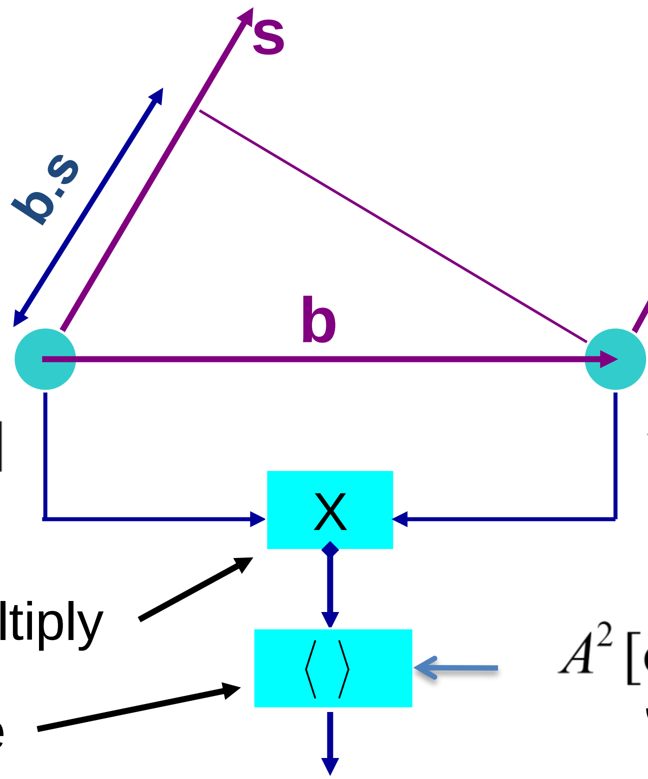
  is formed. This is the 'geometric time delay' – the extra time taken for the signal to reach the more distant sensor.
- Finally, the phase corresponding to this extra distance is defined:

$$\Theta = \omega \tau_g = 2\pi \mathbf{b} \bullet \mathbf{s} / \lambda$$

The Stationary, Quasi-Monochromatic Radio-Frequency Interferometer

Geometric
Time Delay

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$



$$V_1 = E\cos[\omega(t - \tau_g)]$$

$$V_2 = E\cos(\omega t)$$

The path lengths from sensors to multiplier are assumed equal!

multiply

average

$$A^2\,[\underbrace{\cos(\omega\tau_g)}_{\text{Unchanging}} + \underbrace{\cos(2\omega t - \omega\tau_g)}_{\text{Rapidly varying,}}]$$

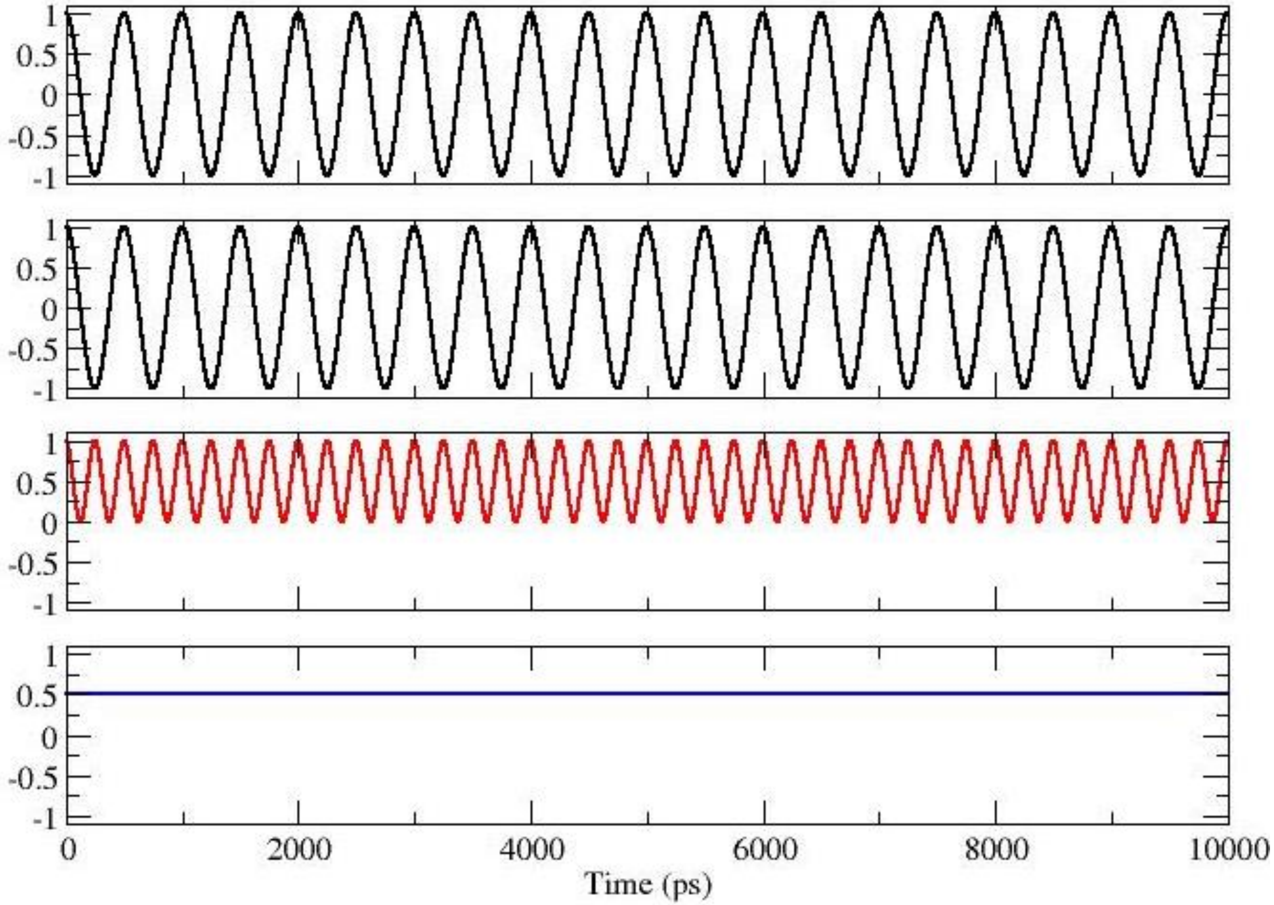
$$R_C = P\cos(\omega\tau_g)$$

Pictorial Example: Signals In Phase

2 GHz Frequency, with voltages in phase:

b.s = nλ, or $\tau_g = n/\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

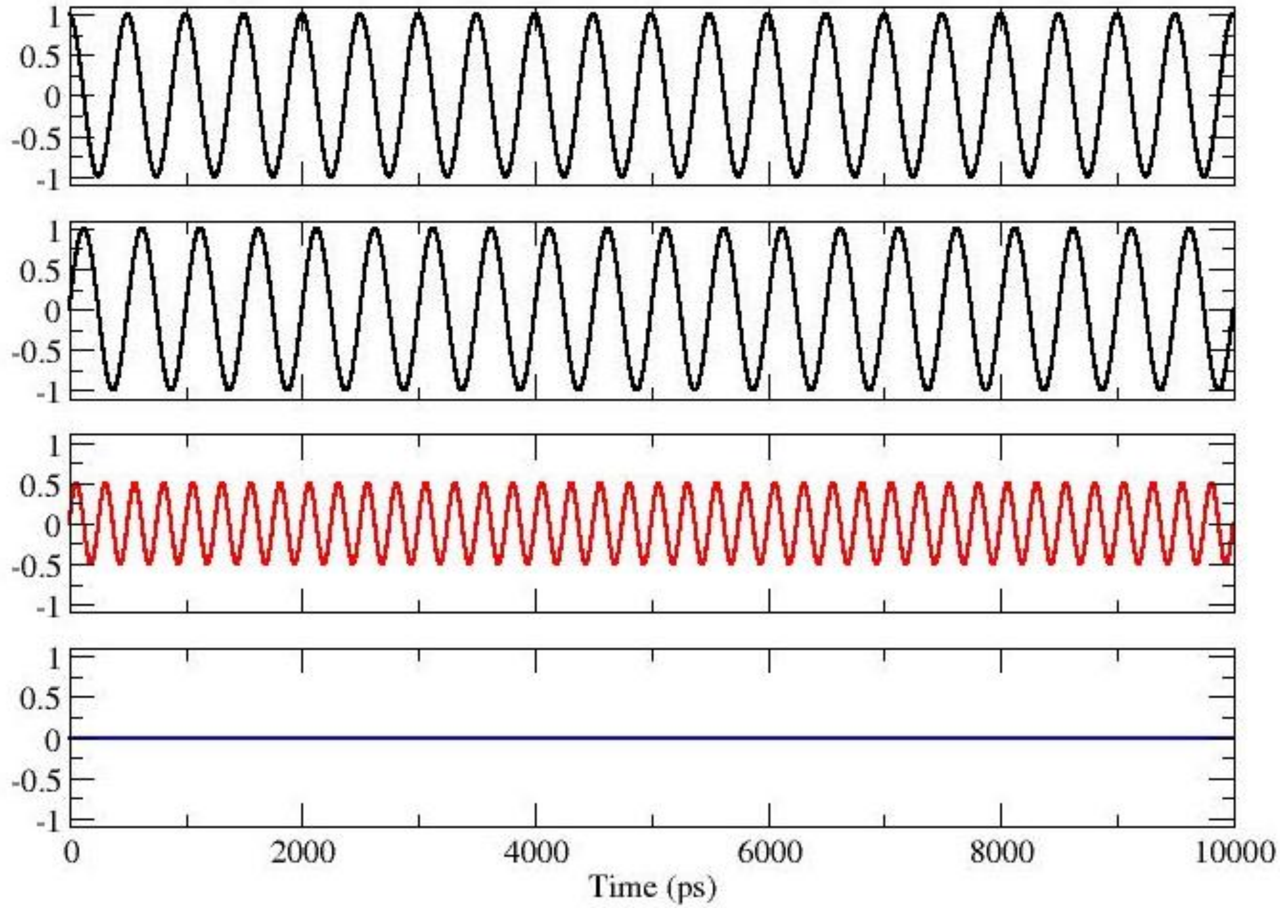


Pictorial Example: Signals in Quad Phase

2 GHz Frequency, with voltages in quadrature phase:

$b.s=(n \pm 1/4)\lambda$, $\tau_g = (4n \pm 1)/4\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

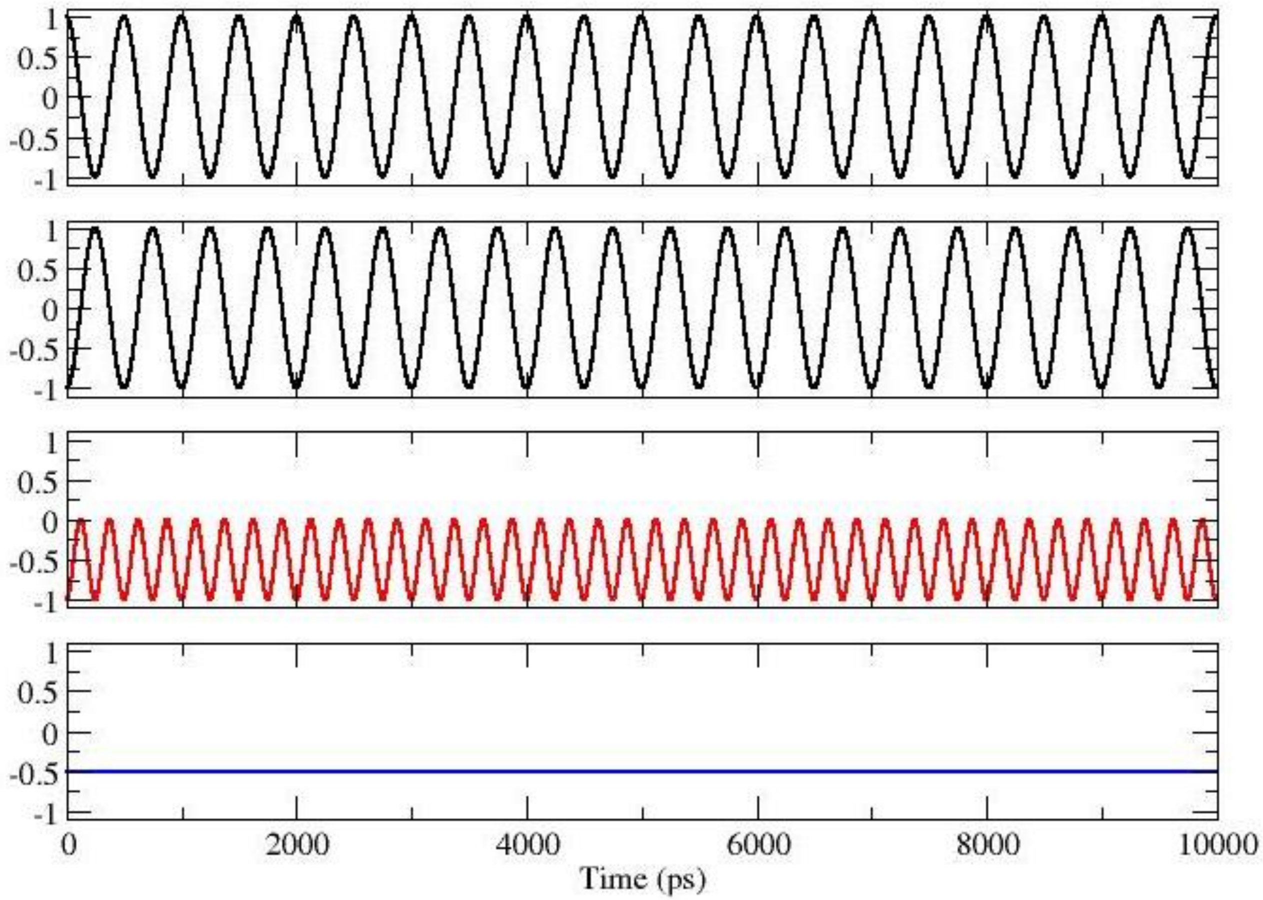


Pictorial Example: Signals out of Phase

2 GHz Frequency, with voltages out of phase:

$b.s=(n \pm \frac{1}{2})\lambda \qquad \tau_g = (2n \pm 1)/2\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



Some General Comments

- The averaged product R_C is dependent on the received power, $P = E^2/2$ and geometric delay, τ_g, and hence on the baseline orientation and source direction:

$$R_C = P\cos(\omega\tau_g) = P\cos\left(2\pi\frac{\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)$$

- Note that R_C is not a a function of:
  - The time of the observation -- provided the source itself is not variable.
  - The location of the baseline -- provided the emission is in the far-field.
  - The actual phase of the incoming signal – the distance of the source does not matter, provided it is in the far-field.

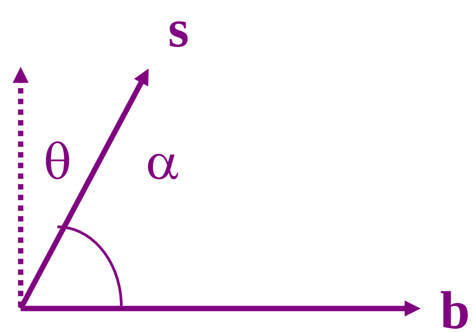
- The strength of the product is dependent on the antenna collecting areas and electronic gains – but these factors can be calibrated for.

Pictorial Illustrations

- To illustrate the response, expand the dot product in one dimension:

$$\frac{\mathbf{b} \bullet \mathbf{s}}{\lambda} = u \cos\alpha = u \sin\theta = ul$$

- Here, **u = b/λ is the baseline length in wavelengths**, and θ is the angle w.r.t. the plane perpendicular to the baseline.
- $l = \cos\alpha = \sin\theta$ is the direction cosine



- Consider the response R_c, as a function of angle, for two different baselines with u = 10, and u = 25 wavelengths:

$$R_C = \cos(20\pi l)$$

Whole-Sky Response

- Top: u = 10

$$R_C = \cos(20\pi l)$$

There are 20 whole fringes over the hemisphere.

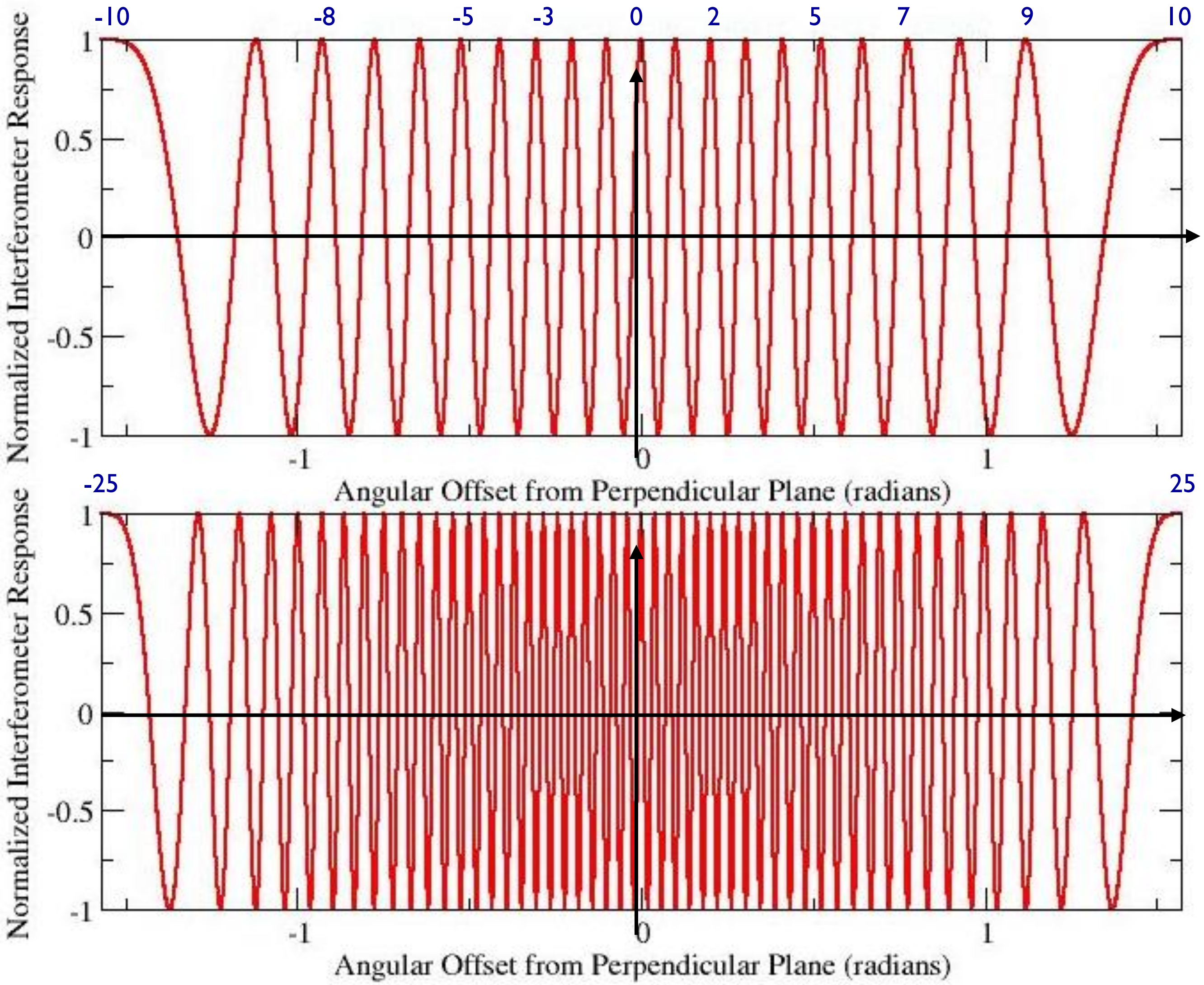
Peak separation 1/10 radians

- Bottom: u = 25

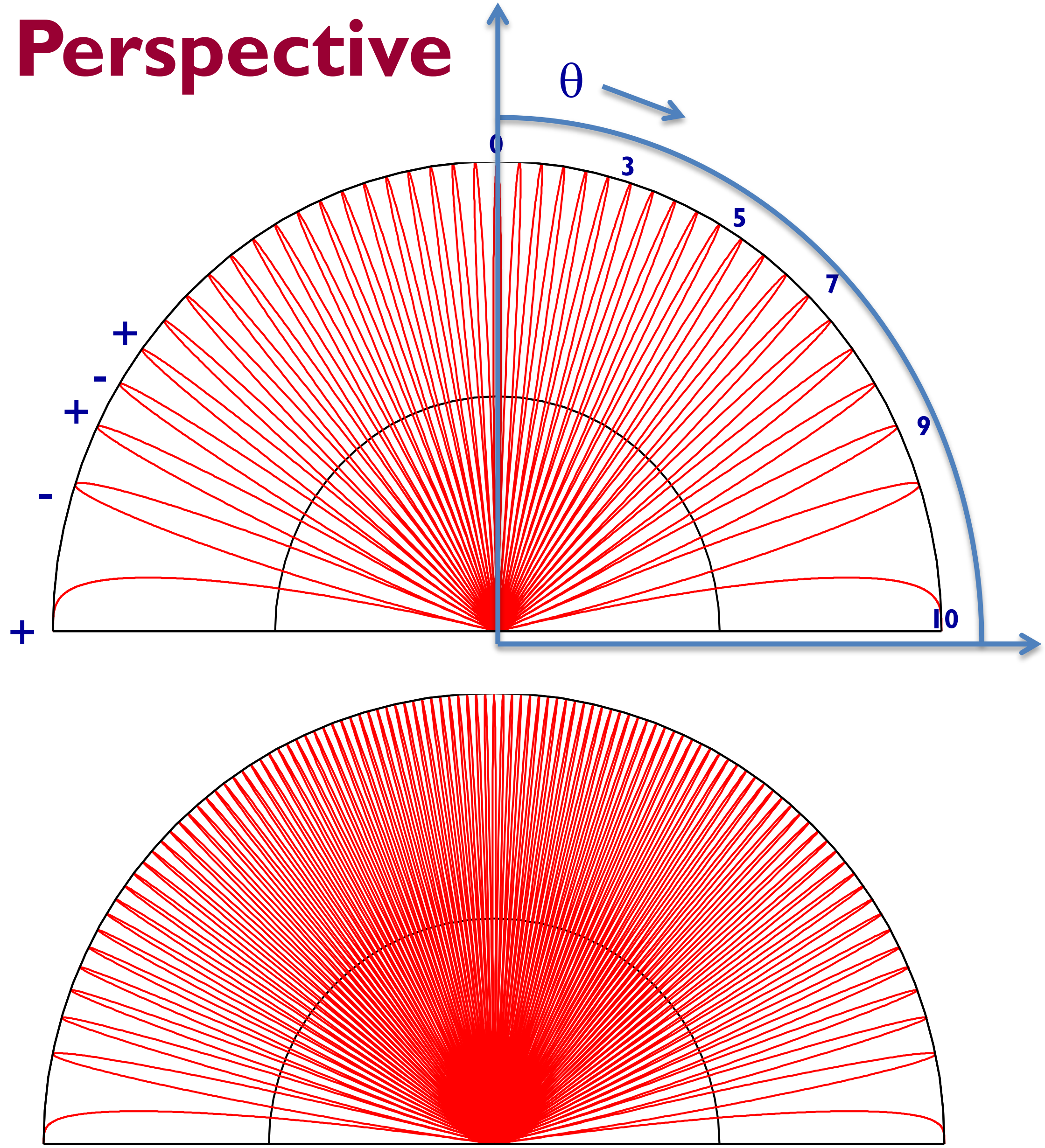
$$R_C = \cos(50\pi l)$$

There are 50 whole fringes over the hemisphere.

Peak separation 1/25 radians.



From an Angular Perspective



Top Panel:

The absolute value of the response for u = 10, as a function of angle.

The ‘lobes’ of the response pattern alternate in sign.

Bottom Panel:

The same, but for u = 25.

Angular separation between lobes (of the same sign) is

$\delta\theta \sim 1/u = \lambda/b$ radians.

Hemispheric Pattern

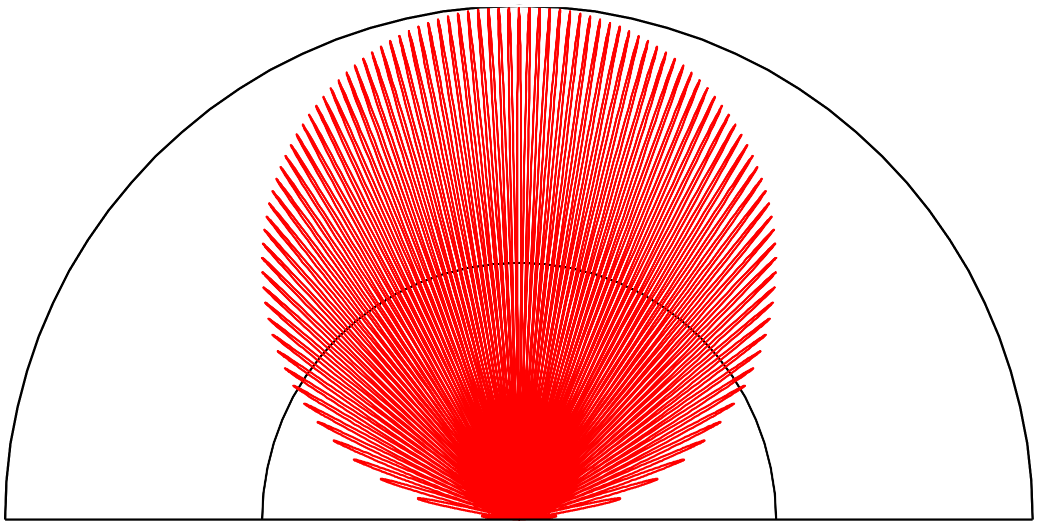
- The preceding plot is a meridional cut through the hemisphere, oriented along the baseline vector.
- In the two-dimensional space, the fringe pattern consists of a series of coaxial cones, oriented along the baseline vector.
- The figure is a two-dimensional representation when u = 4.
- As viewed along the baseline vector, the fringes show a 'bulls-eye' pattern – concentric circles.

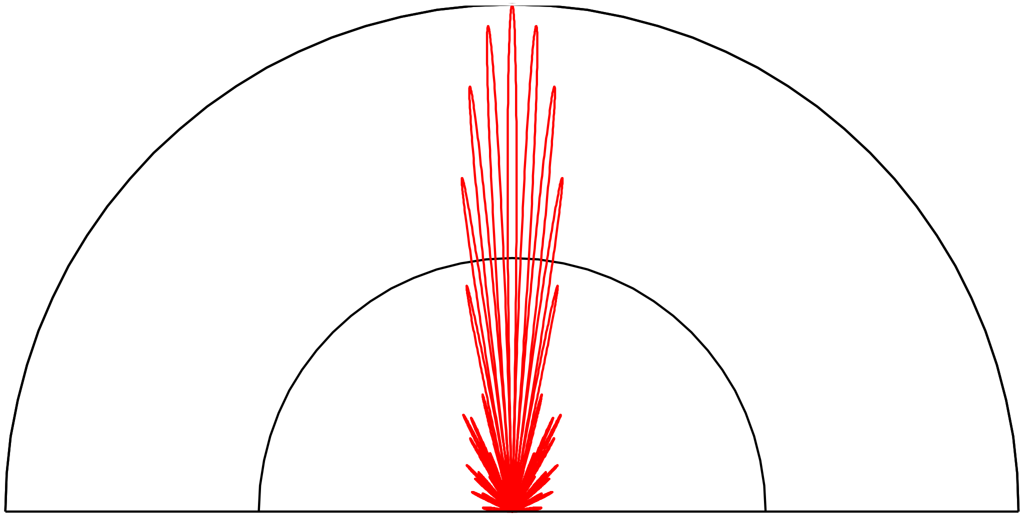
The Effect of the Sensor

- The patterns shown presume the sensor (antenna) has isotropic response.
- This is a convenient assumption, but doesn't represent reality.
- Real sensors impose their own patterns, which modulate the amplitude and phase, of the output.
- Large antennas have very high directivity -- very useful for some applications.

The Effect of Sensor Patterns

- Sensors (or antennas) are not isotropic, and have their own responses.
- **Top Panel:** The interferometer pattern with a cos(θ)-like sensor response.
- **Bottom Panel:** A multiple-wavelength aperture antenna has a narrow beam, but also sidelobes.





The Response from an Extended Source

- The response from an extended source is obtained by summing the responses at each antenna to all the emission over the sky, multiplying the two, and averaging:

$$R_C = \left\langle \iint V_1 d\Omega_1 \times \iint V_2 d\Omega_2 \right\rangle$$

- The averaging and integrals can be interchanged and, **providing the emission is spatially incoherent**, we get

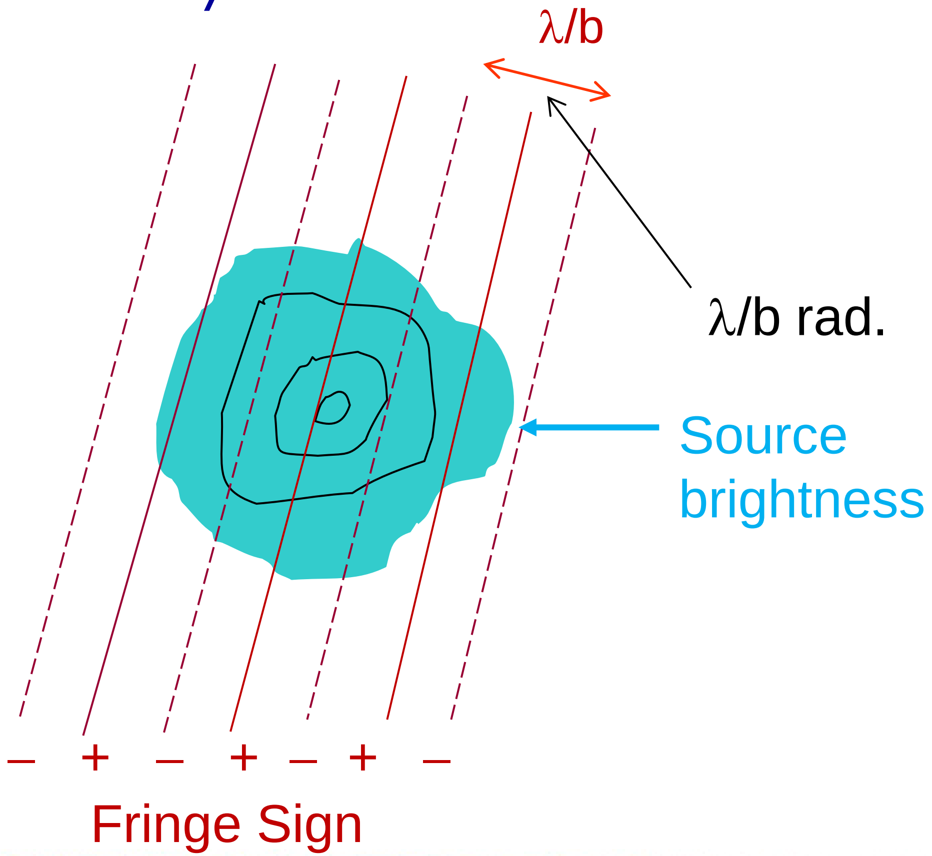
$$R_C = \iint I_\nu(\mathbf{s}) \cos(2\pi\nu \mathbf{b}\cdot\mathbf{s}/c)\, d\Omega$$

- This expression links what we want – the source brightness on the sky, $I_\nu(\mathbf{s})$, – to something we can measure - R_C, the interferometer response.
- Can we recover $I_\nu(\mathbf{s})$ from observations of R_C?

A Schematic Illustration in 2-D

- The correlator can be thought of 'casting' a cosinusoidal coherence pattern, of angular scale ~λ/b radians, onto the sky.
- The correlator multiplies the source brightness by this coherence pattern, and integrates (sums) the result over the sky.

- Orientation set by baseline geometry.
- Fringe separation set by (projected) baseline length and wavelength.
  - Long baseline gives close-packed fringes
  - Short baseline gives widely-separated fringes
- Physical location of baseline unimportant, provided source is in the far field.



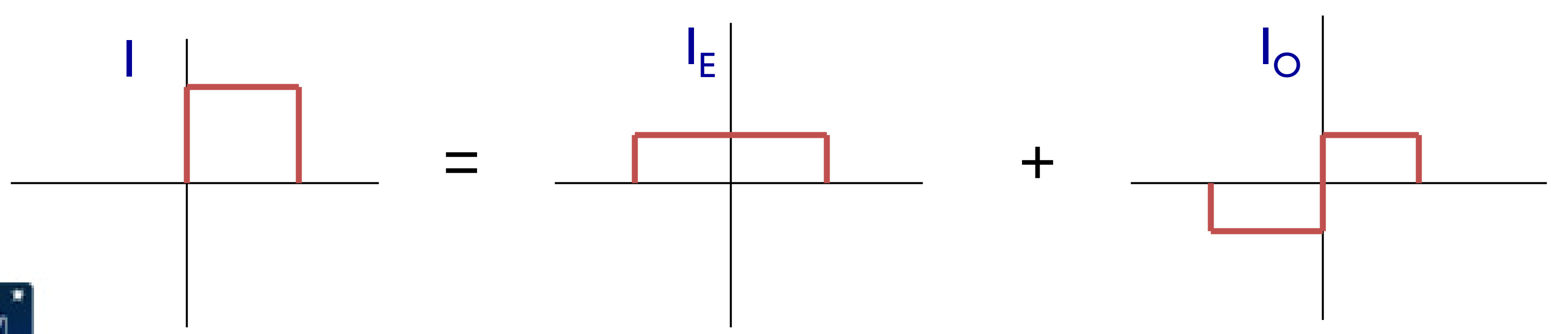
A Short Mathematics Digression – Odd and Even Functions

- Any real function, I(x,y), can be expressed as the sum of two real functions which have specific symmetries:

$$I(x,y) = I_E(x,y) + I_O(x,y)$$

An even part: $$I_E(x,y) = \frac{I(x,y) + I(-x,-y)}{2} = I_E(-x,-y)$$

An odd part: $$I_O(x,y) = \frac{I(x,y) - I(-x,-y)}{2} = -I_O(-x,-y)$$



Why One Correlator is Not Enough

- The correlator response, R_c:

$$R_C = \iint I_\nu(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

is not enough to recover the correct brightness. Why?

- Only the even part of the distribution is seen.
- Suppose that the source of emission has a component with odd symmetry:

$$I_o(\mathbf{s}) = -I_o(-\mathbf{s})$$

- Since the cosine fringe pattern is even, the response of our interferometer to the odd brightness distribution is 0.

$$R_C = \iint I_o(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = 0$$

- Hence, we need more information if we are to completely recover the source brightness.

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Why Two Correlations are Needed

- The integration of the cosine response, R_c, over the source brightness is sensitive to only the even part of the brightness:

$$R_C = \iint I(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega = \iint I_E(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega$$

since the integral of an odd function (I_O) with an even function (cos x) is zero.

- To recover the 'odd' part of the brightness, I_O, we need an 'odd' fringe pattern. Let us replace the 'cos' with 'sin' in the integral

$$R_s = \iint I(\mathbf{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega = \iint I_o(\mathrm{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega$$

since the integral of an even times an odd function is zero.

- To obtain this necessary component, we must make a 'sine' pattern. How?

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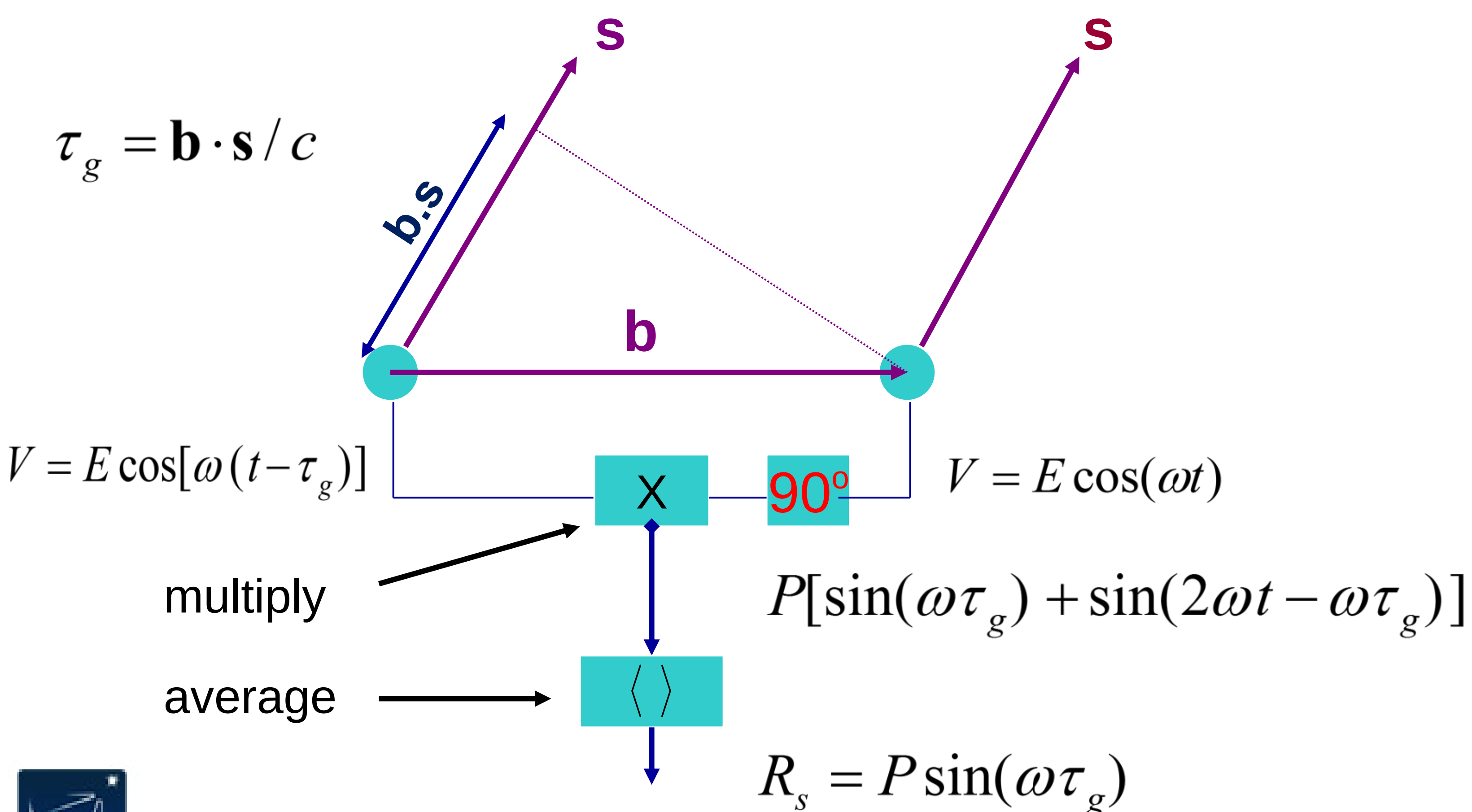
Making a SIN Correlator

- We generate the 'sine' pattern by inserting a 90 degree phase shift in one of the signal paths.



Define the Complex Visibility

- We now DEFINE a complex function, the complex visibility, V, from the two independent (real) correlator outputs R_C and R_S:

$$V = R_C - iR_S = Ae^{-i\phi}$$

where

$$A = \sqrt{R_C^2 + R_S^2}$$

$$\phi = \tan^{-1}\left(\frac{R_S}{R_C}\right)$$

- This gives us a beautiful and useful relationship between the source brightness, and the response of an interferometer:

$$V_\upsilon(\mathbf{b}) = R_C - iR_S = \iint I_\nu(s)\, e^{-2\pi i\nu \mathbf{b}\cdot\mathbf{s}/c}\, d\Omega$$

- With the right geometry, this is a 2-D Fourier transform, giving us a well established way to recover $I(\mathbf{s})$ from $V(\mathbf{b})$.

The Complex Correlator and Complex Notation

- A correlator which produces both 'Real' and 'Imaginary' parts – or the Cosine and Sine fringes, is called a 'Complex Correlator'
  - For a complex correlator, think of two independent sets of projected sinusoids, 90 degrees apart on the sky.
  - In our scenario, both components are necessary, because we have assumed there is no motion – the 'fringes' are fixed on the source emission, which is itself stationary.
- The complex output of the complex correlator also means we can use complex analysis throughout: Let:

$$V_1 = A\cos(\omega t) = \mathrm{Re}\left(Ae^{-i\omega t}\right)$$

$$V_2 = A\cos[\omega(t - \mathbf{b}\cdot\mathbf{s}/c)] = \mathrm{Re}\left(Ae^{-i\omega(t-\mathbf{b}\bullet\mathbf{s}/c)}\right)$$

- Then:

$$P_{corr} = \left\langle V_1 V_2^* \right\rangle = A^2 e^{-i\omega\,\mathbf{b}\bullet\mathbf{s}/c}$$

Wideband Phase Shifters – Hilbert Transform

- For a quasi-monochromatic signal, forming a the 90 degree phase shift to the signal path is easy --- add a piece of cable $\lambda/4$ wavelengths long.
- For a wideband system, this obviously won't work.
- In general, a wideband device which phase shifts each spectral component by 90 degrees, while leaving the amplitude intact, is a Hilbert Transform.
- For real interferometers, such an operation can be performed by analog devices.
- Far more commonly, this is done using digital techniques.
- The complex function formed by a real function and its Hilbert transform is termed the 'analytic signal'.

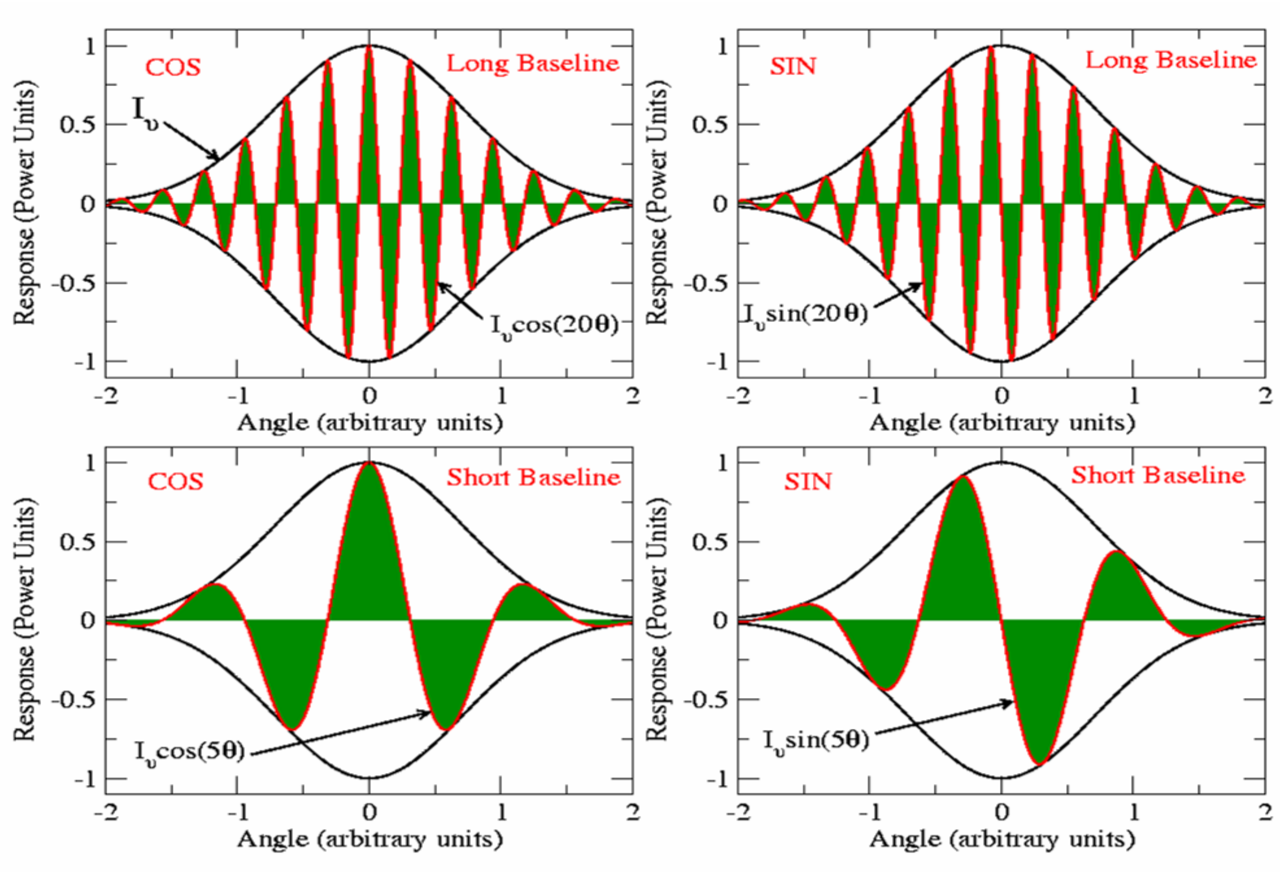
Picturing the Visibility

- The source brightness is Gaussian, shown in black.
- The interferometer 'fringes' are in red.
- The visibility is the integral of the product – the net dark green area.

R_C R_S

Long Baseline

Short Baseline

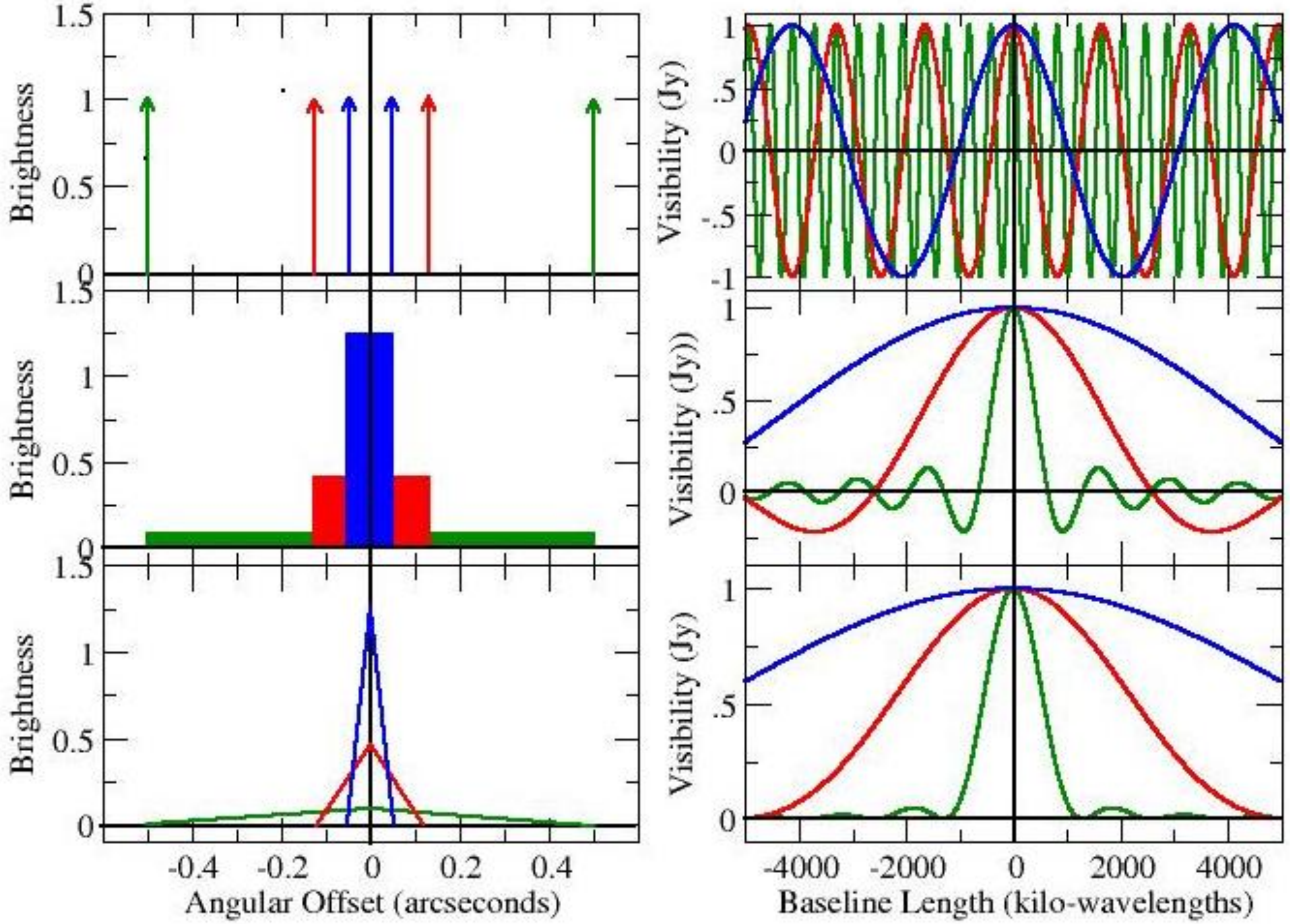


Examples of 1-Dimensional Visibilities

- Simple pictures are easy to make illustrating 1-dimensional visibilities.

Brightness Distribution | Visibility Function

- Unresolved Doubles
- Uniform
- Central Peaked

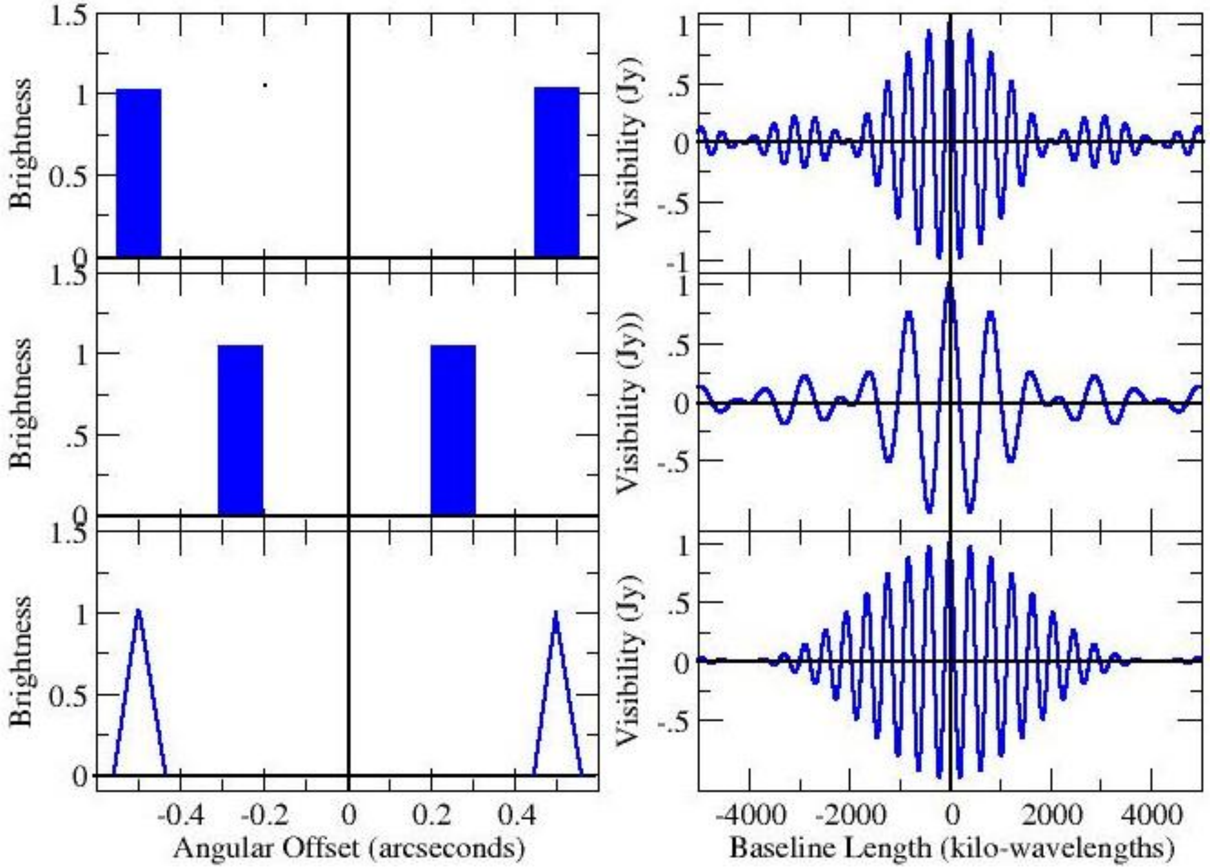


More Examples

- Simple pictures are easy to make illustrating 1-dimensional visibilities.

Brightness Distribution | Visibility Function

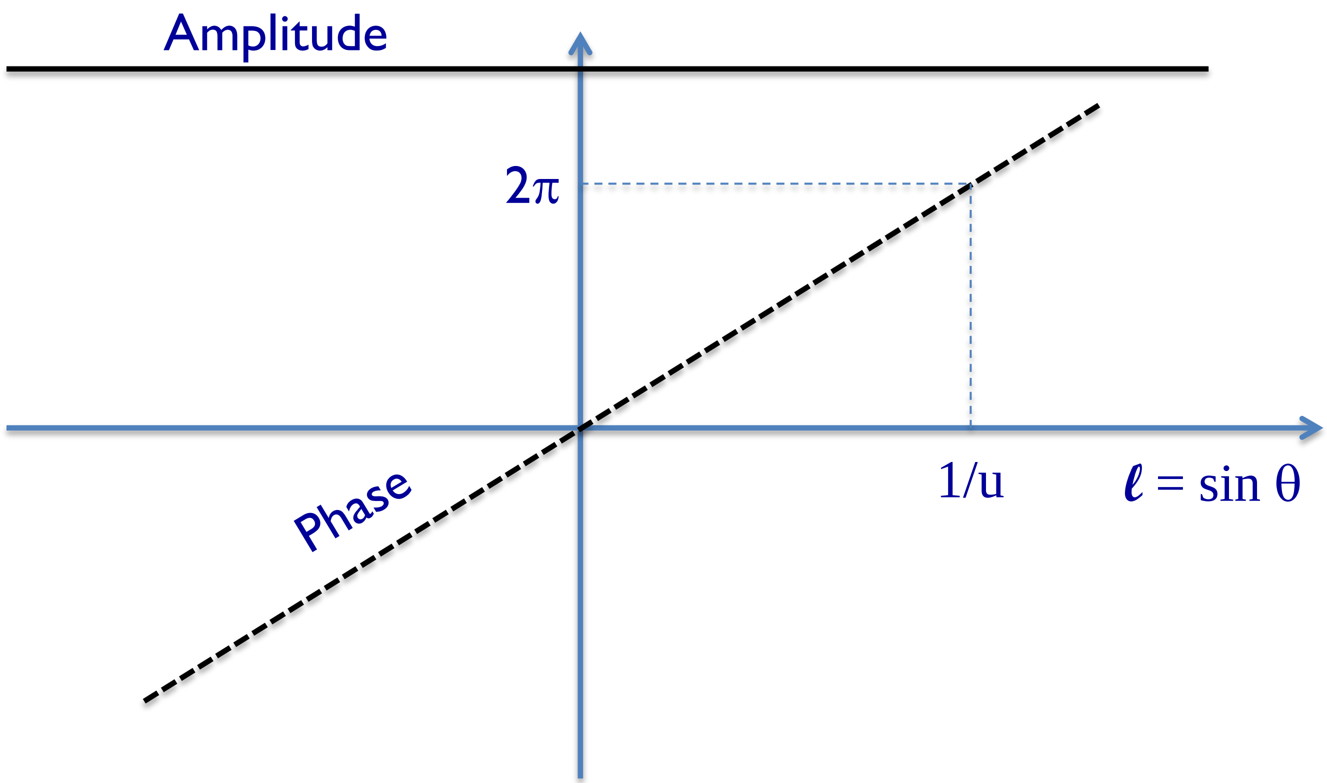
- Resolved Double
- Resolved Double
- Central Peaked Double



Another Way to Conceptualize …

- For those of you adept in thinking in terms of complex functions, another way to picture the effect of the interferometer may be attractive …
- The interferometer casts a *phase slope* across the (real) brightness distribution.
  - The phase slope becomes steeper for longer baselines, or higher frequencies, and is zero for zero baseline.
  - The phase is zero at the phase origin.
  - The amplitude response is unity (ignoring the primary beam) throughout.
- The Visibility is the complex integral of the brightness times the phase ramp.

The Complex Integral



Basic Characteristics of the Visibility

- For a zero-spacing interferometer, we get the 'single-dish' (total-power) response.
- As the baseline gets longer, the visibility amplitude will in general decline.
- When the visibility is close to zero, the source is said to be 'resolved out'.
- Interchanging antennas in a baseline causes the phase to be negated – the visibility of the 'reversed baseline' is the complex conjugate of the original. (Why?)
- Mathematically, the visibility is Hermitian. ($V(u) = V^*(-u)$).

Some Comments on Visibilities

- The Visibility is a unique function of the source brightness.
- The two functions are related through a Fourier transform. $V_{\nu}(u,v) \Leftrightarrow I(l,m)$
- An interferometer, at any one time, makes one measure of the visibility, at baseline coordinate (u,v).
- `Sufficient knowledge' of the visibility function (as derived from an interferometer) will provide us a `reasonable estimate' of the source brightness.
- How many is 'sufficient', and how good is 'reasonable'?
- These simple questions do not have easy answers…