

A Semi-Lagrangian Reduced Order Modeling Framework for Advection-Dominated Systems

Scalable Algorithms to Build Surrogates for Astrophysical Flows

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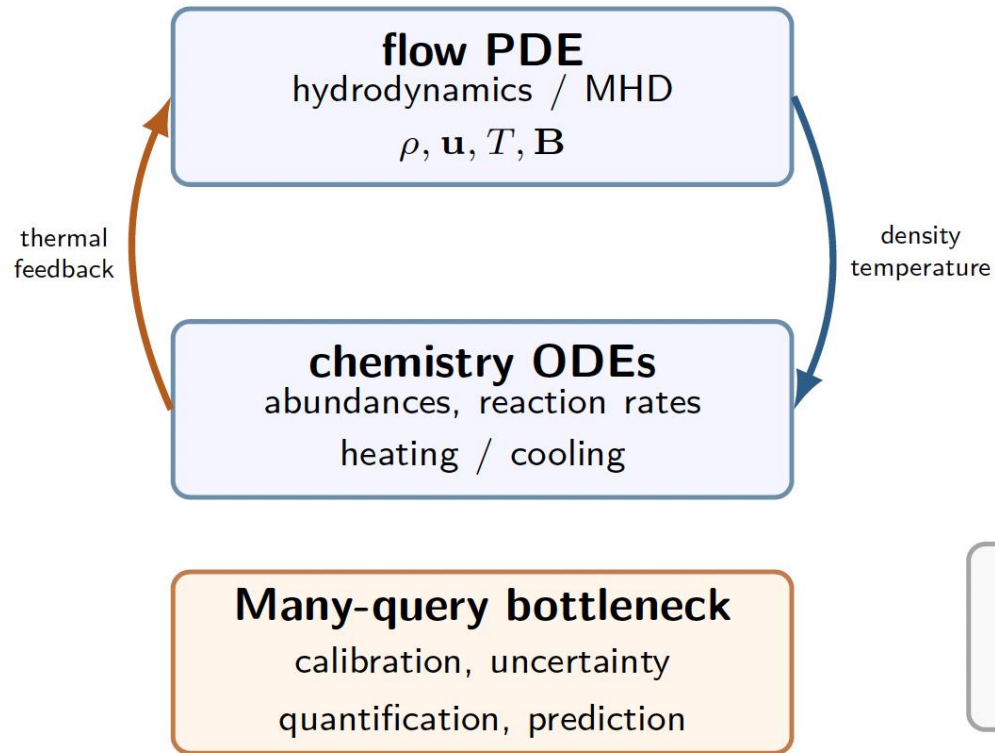
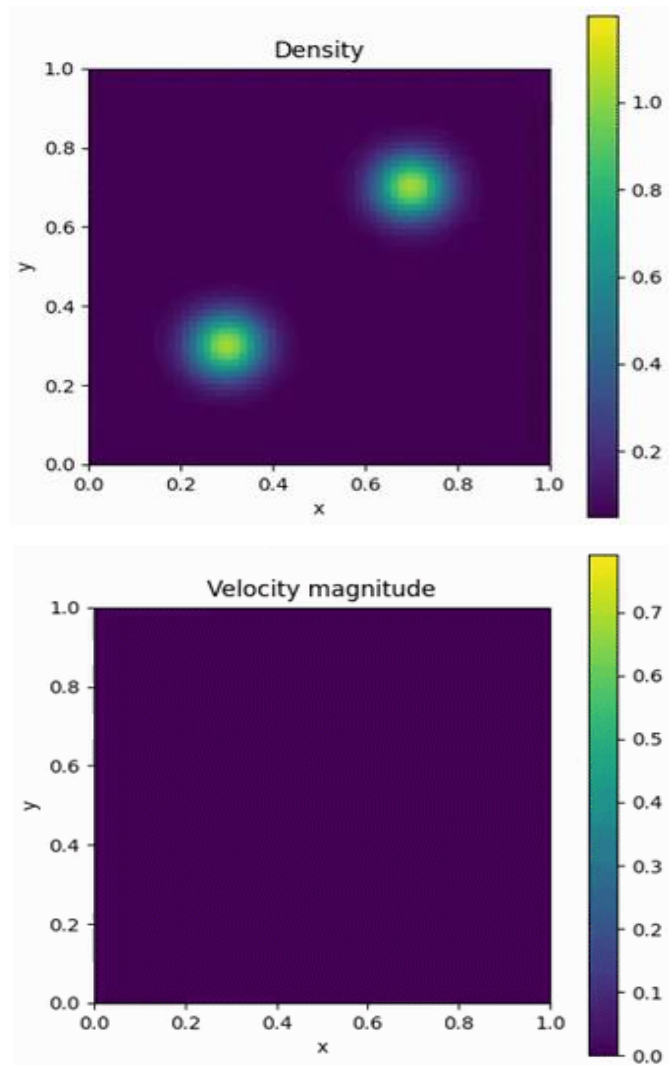
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Disclaimer: *This work does not involve any Neural Networks.
It is based on old-school Numerical Analysis.*



The Real Cost Is Repeated Simulation

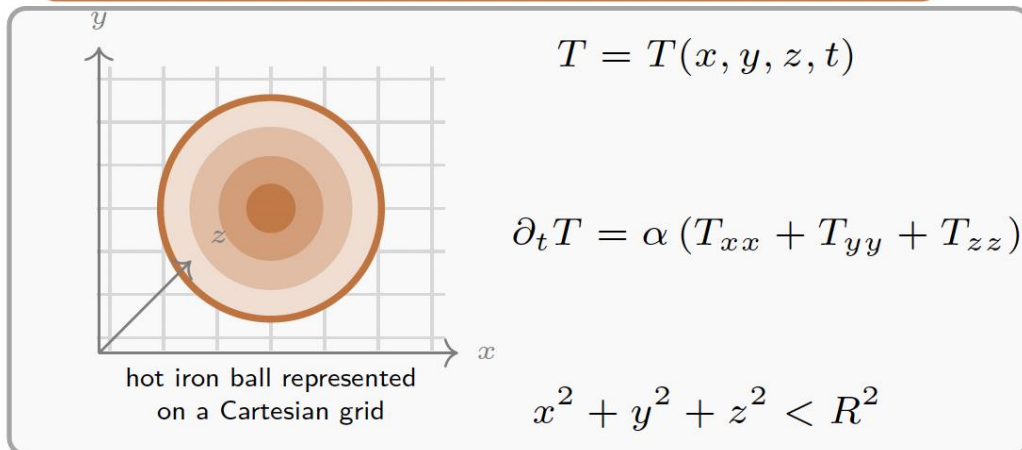


Can we preserve the transported physics at a fraction of the repeated cost?

A Surrogate Is an Effective Coordinate System

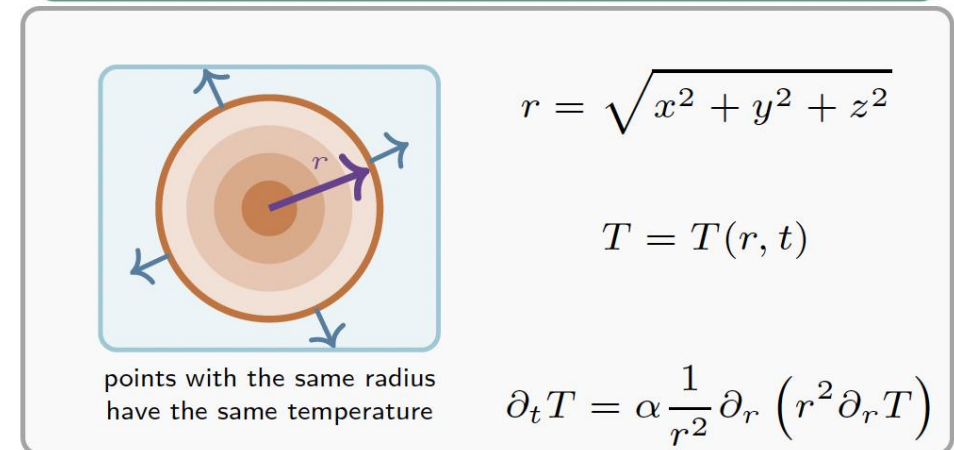
A good surrogate does not merely discard variables: it finds coordinates in which the important dynamics is simpler.

Cartesian coordinates: valid, but the symmetry is hidden



$\Rightarrow$
use the
symmetry

Spherical coordinates: the dynamics becomes radial



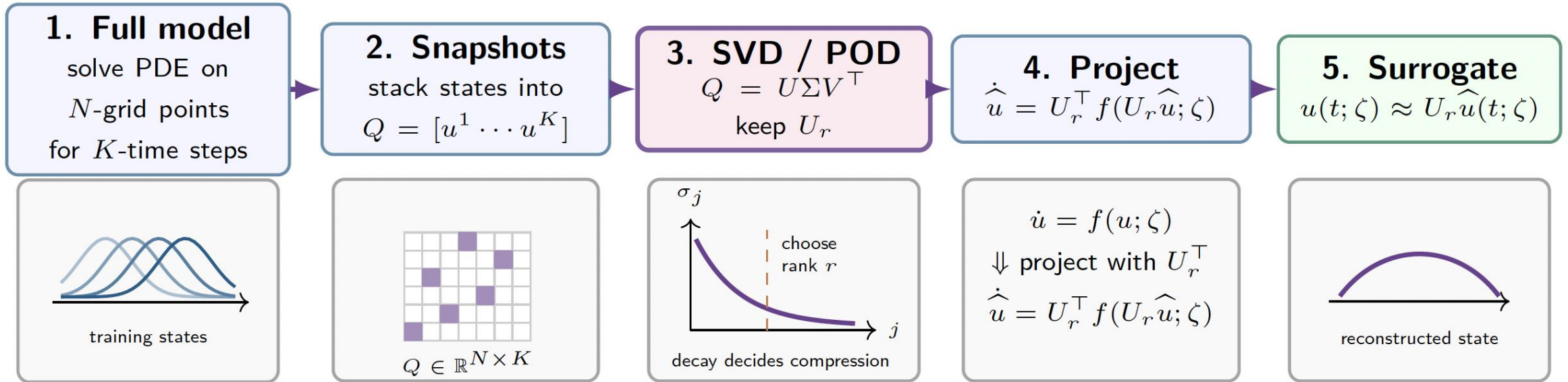
Three spatial coordinates; the spherical structure is not explicit.

One spatial coordinate; the symmetry makes the dynamics intuitive.

Surrogate viewpoint: find an effective coordinate vector $z(t) \in \mathbb{R}^r$ and basis $\Phi \in \mathbb{R}^{N \times r}$ such that

$$u(t) \approx \Phi z(t), \quad r \ll N.$$

Global POD–Galerkin: One Basis for the Whole Movie



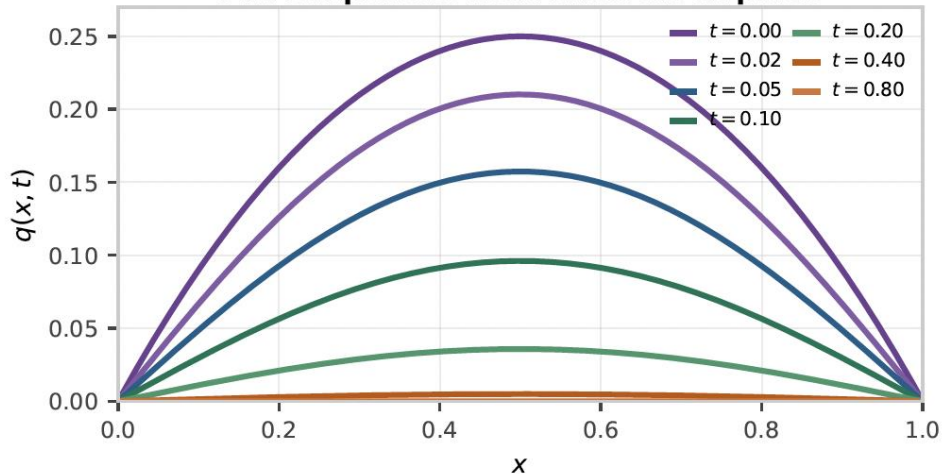
Assumption: the whole trajectory is close to one fixed low-dimensional subspace.
Successful "IF": rapid singular-value decay $\Rightarrow$ When a small no. of basis captures most snapshot energy.

Diffusion Example: Exponential Singular-Value Decay

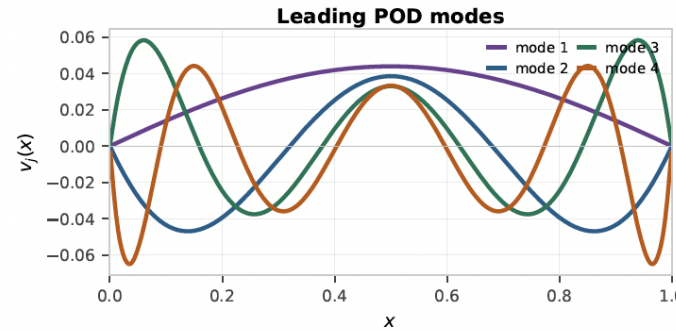
Cooling rod

$$\begin{aligned} q_t &= q_{xx}, \quad x \in [0, 1], \\ q(0, t) &= q(1, t) = 0 \\ q(x, 0) &= x(1 - x) \end{aligned}$$

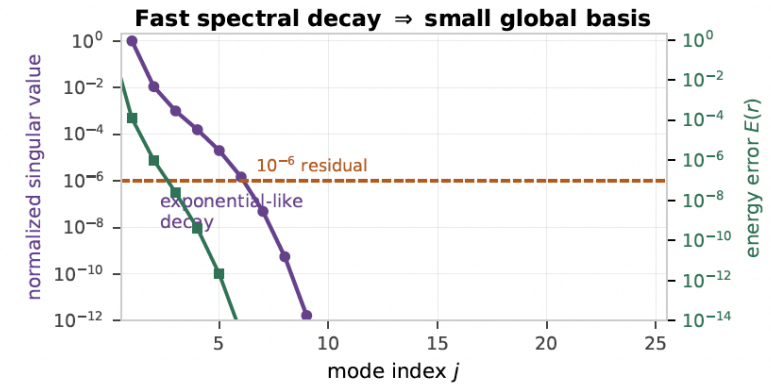
Full snapshots: heat smooths in place



POD-friendly geometry: profile stays centered; heat decays and smooths.



Few global modes
diffusion removes high-frequency content quickly



POD-Galerkin

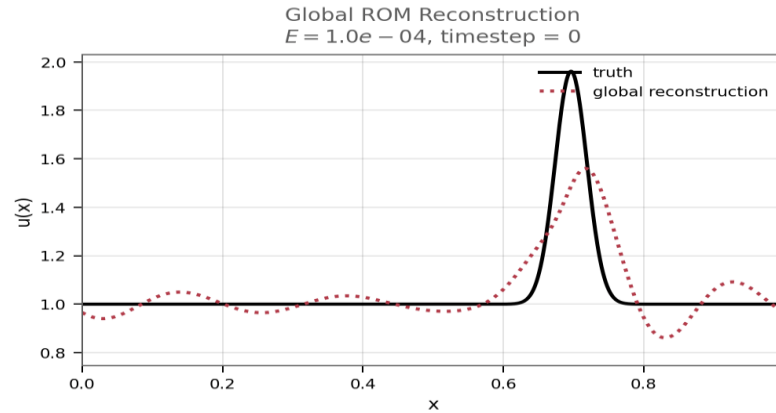
$$Q = U \Sigma V^T, \quad q \approx U_r \hat{q}$$

$$\dot{\hat{q}} = U_r^T A U_r \hat{q}$$

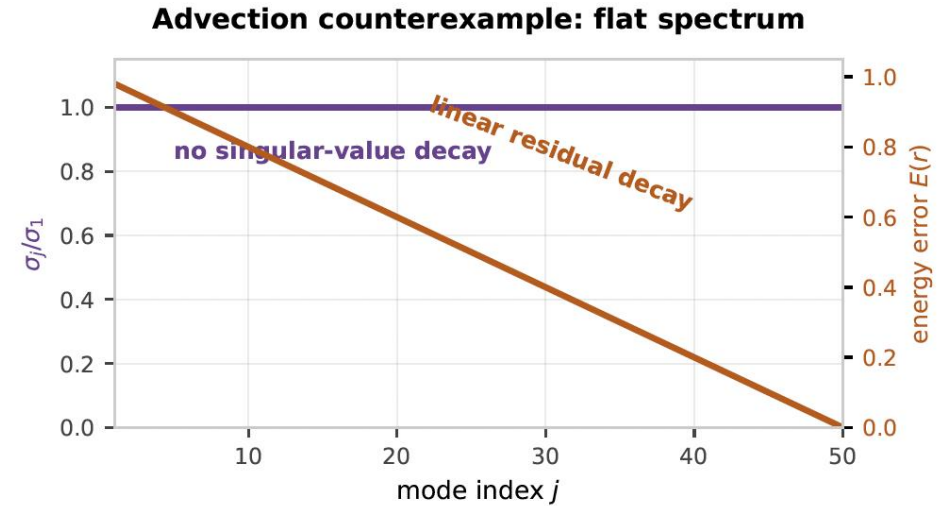
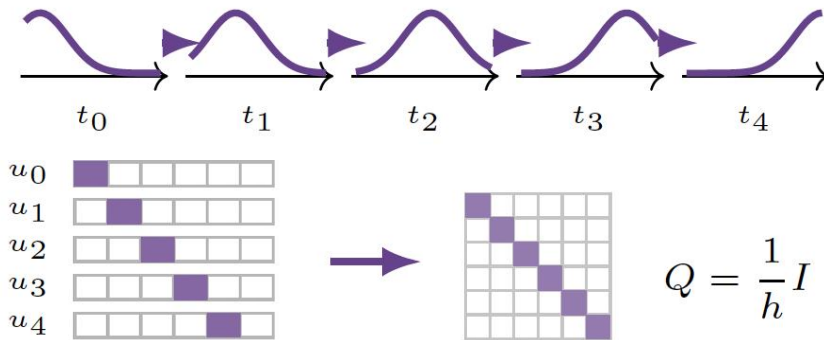
$$E(r) = \frac{\sum_{j>r} \sigma_j^2}{\sum_{j \geq 1} \sigma_j^2} \quad (\text{Energy error})$$

Exponential decay $\Rightarrow$ small r .

Advection Example: Flat Spectrum and Linear Energy Error



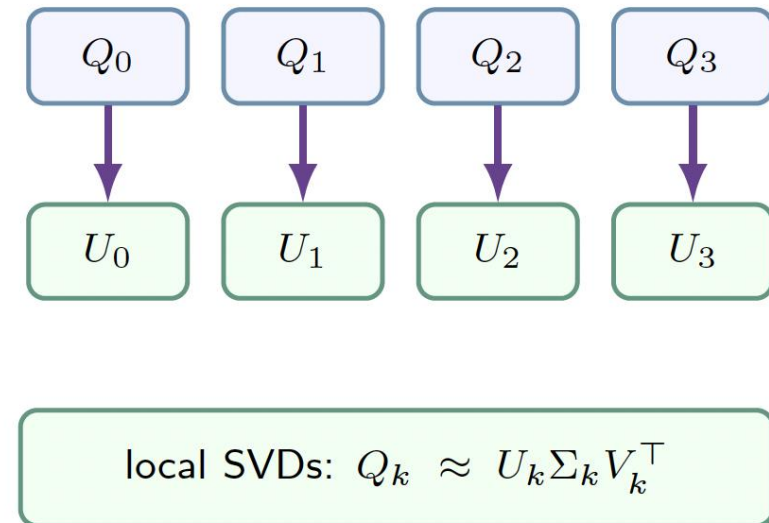
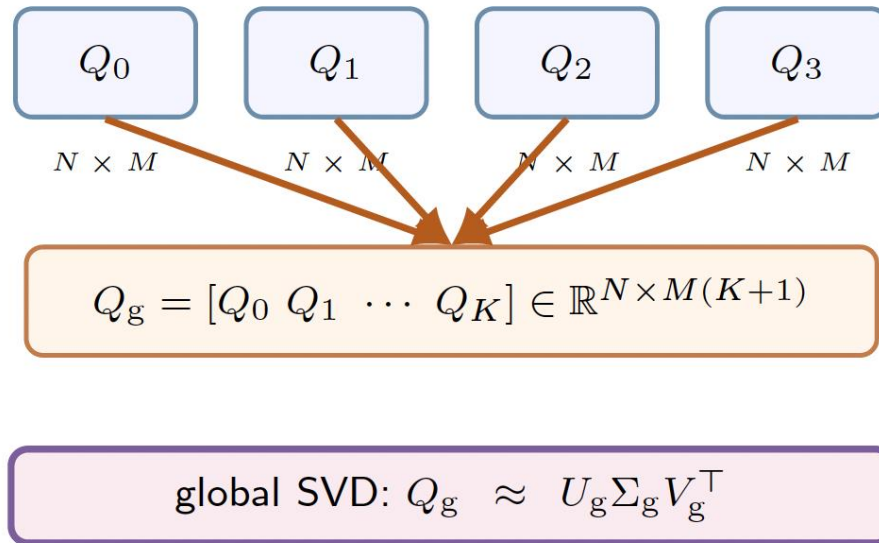
One structure changing location
 $u_t + a(x, t)u_x = 0, x \in [0, 1],$
 $u(0, t) = q(1, t)$ (periodic)
 $u(x, 0) = u_0(x)$



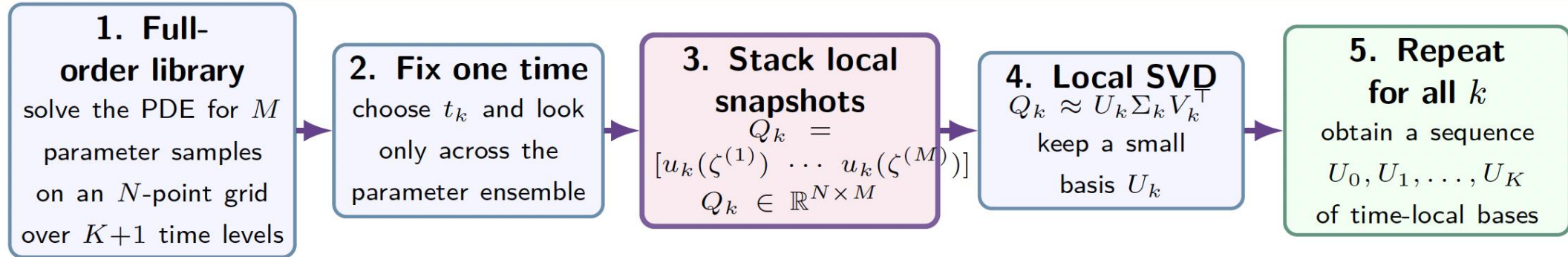
$u_k = \frac{1}{h} e_k, \quad Q = [u_0 \ u_1 \ \cdots \ u_{N-1}] = \frac{1}{h} I.$
Singular values: $\sigma_1 = \cdots = \sigma_N$ and
Energy error: $E(r) = \frac{\sum_{j>r} \sigma_j^2}{\sum_{j \geq 1} \sigma_j^2} = 1 - \frac{r}{N}$

Energy error decreases only **linearly**.
 To make $E(r)$ small, the
 global rank must scale like N .

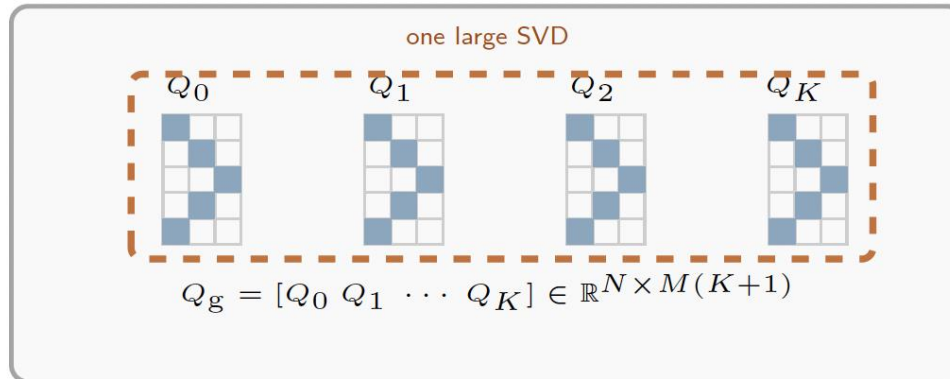
Global vs. Local Basis Approaches



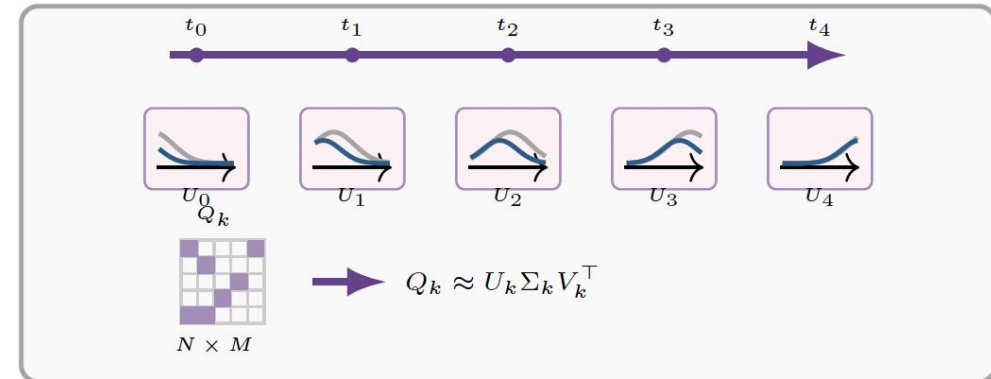
First Reduction: Use a Sequence of Local Basis



Global POD stacks *all* times and *all* parameters before one big SVD



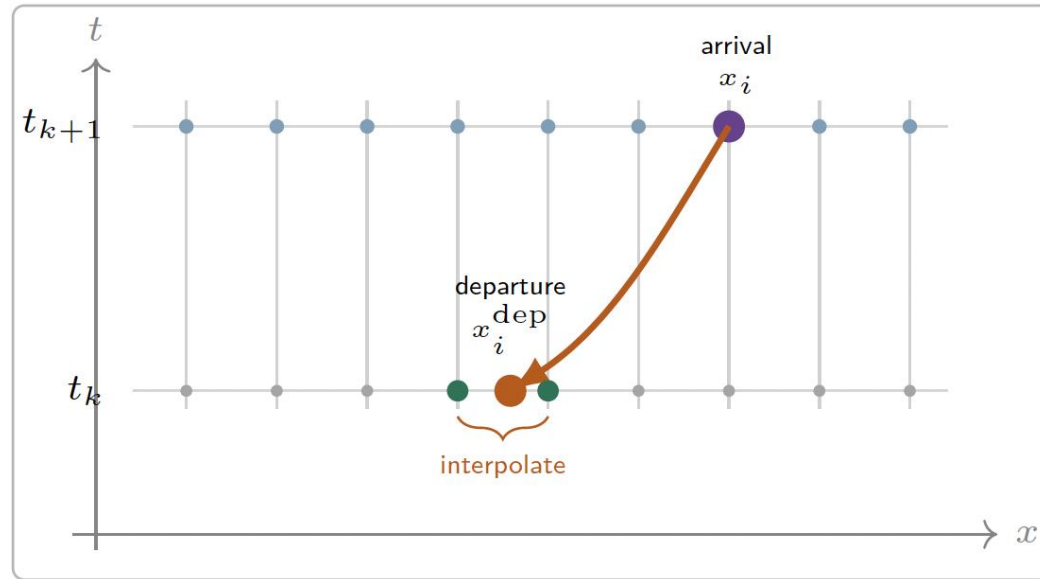
Local POD fixes t_k and stacks only across parameters in the DOF-parameter space



Key idea: many small SVDs of size $N \times M$ replace one giant SVD of size $N \times M \times (K+1)$.

Semi-Lagrangian Transport "Advects" the Local Basis

For an arrival grid point at t_{k+1} , trace **backward** to its departure point at t_k .



$$u_{k+1}(x_i) \approx u_k(x_i^{\text{dep}})$$

Reduced update

$$\hat{u}_{k+1} = U_{k+1}^\top \mathcal{T}_k(\zeta^*) U_k \hat{u}_k.$$

Eulerian

$$u_t + a(x, \zeta) u_x = 0$$

fixed storage grid

Lagrangian

$$\frac{dX}{dt} = a(X, \zeta), \quad \frac{du}{dt} = 0$$

follow characteristics

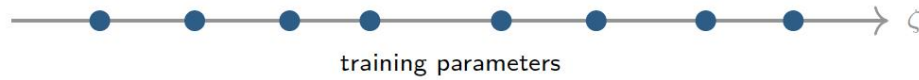
Semi-Lagrangian

$$u(x_i, t_{k+1}) \approx u(X(t_k; t_{k+1}, x_i), t_k)$$

backward tracing + fixed grid

Train Once, Predict an Unseen Flow

Offline: build local spaces
from training simulations



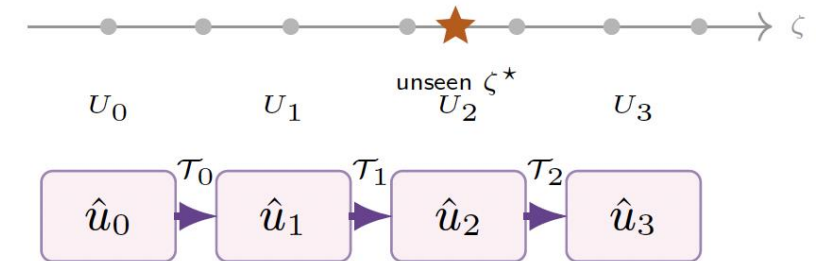
FOM solves

$Q_0, \dots, Q_K$

$U_0, \dots, U_K$

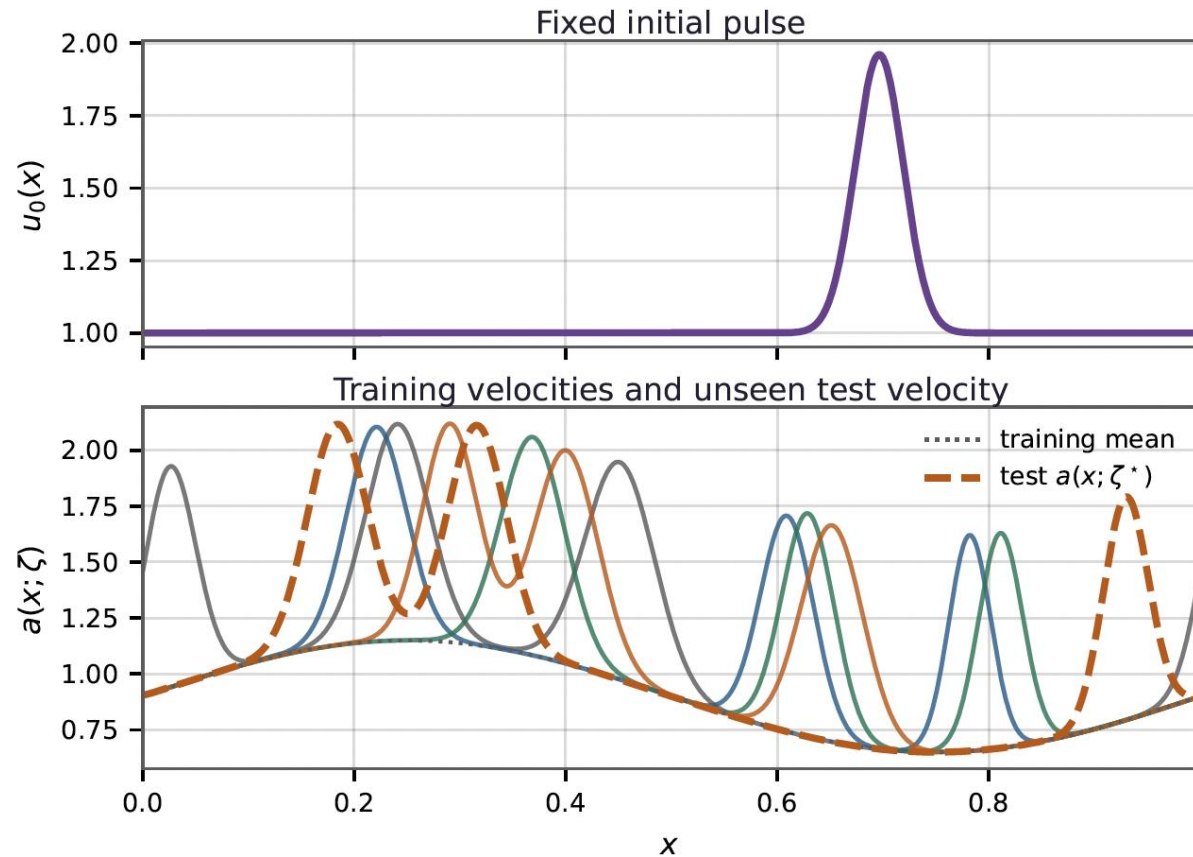
local bases stored once
for many future queries

Online: march through local spaces



$$\hat{u}_{k+1} = U_{k+1}^\top \mathcal{T}_k(\zeta^*) U_k \hat{u}_k$$

Numerical Experiments: Problem statement



Model problem

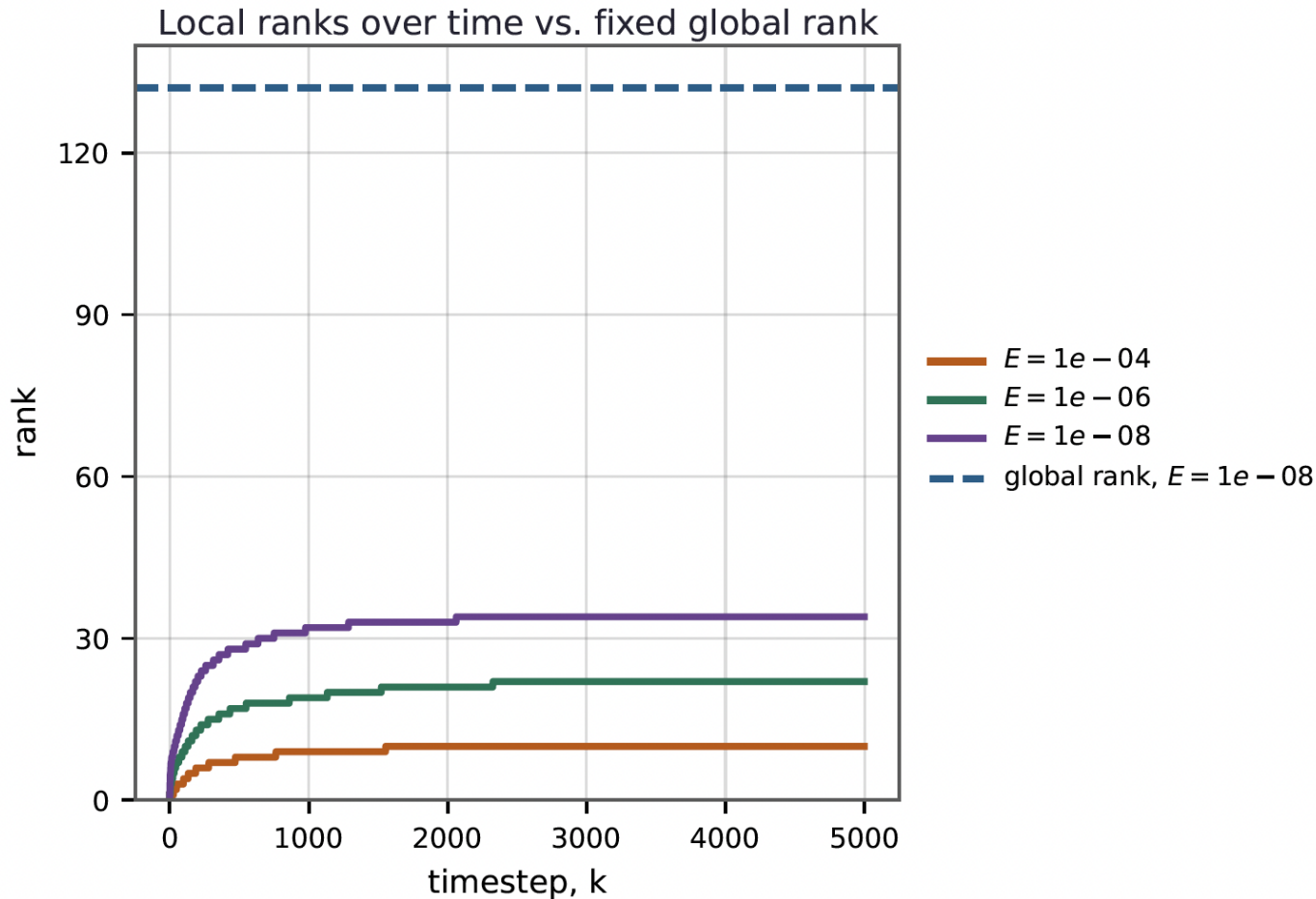
$$u_t + a(x; \zeta)u_x = 0, \quad x \in [0, 1),$$
$$u(0, t; \zeta) = u(1, t; \zeta).$$

Training: many samples of $a(x; \zeta)$

Testing: one unseen velocity $a(x; \zeta^*)$

Question: can local bases
and SL transport generalize?

Local Spaces Remove Most of the Apparent Rank



Energy error	Max local rank	Global rank
10^{-4}	10	36
10^{-5}	16	55
10^{-6}	22	77
10^{-7}	28	104
10^{-8}	34	132

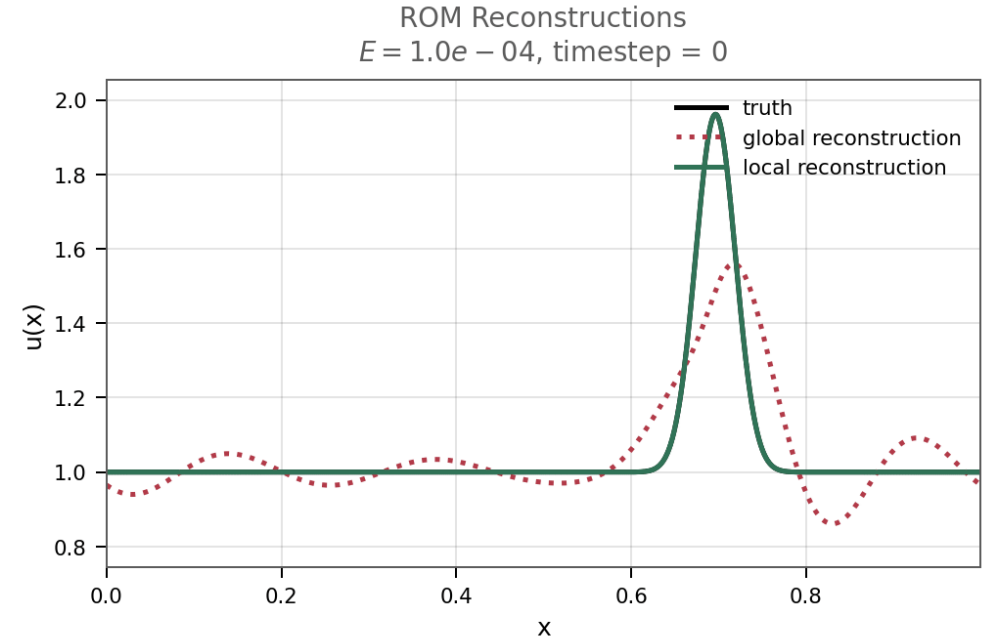
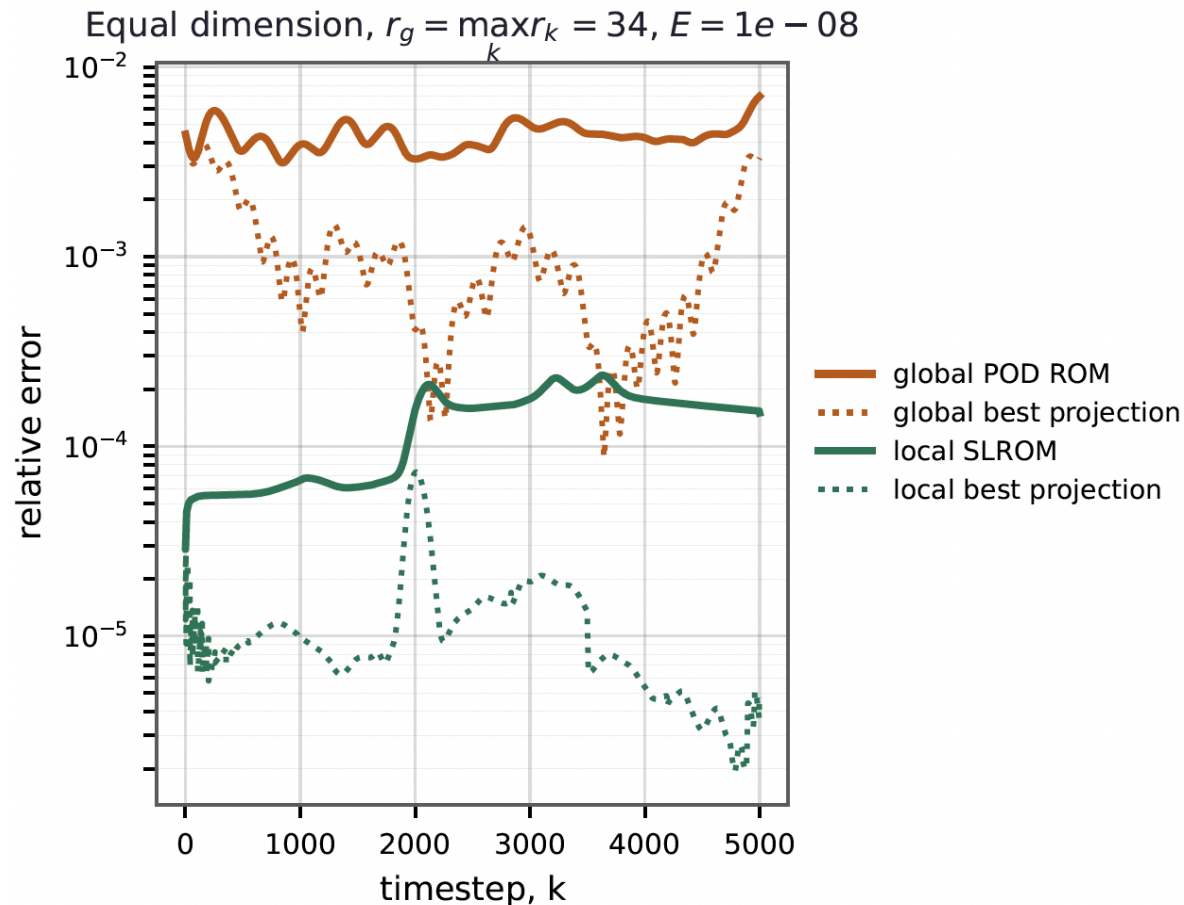
Error Definition

For singular values $\{\sigma_j\}$, the rule for energy error,

$$E = 1 - \frac{\sum_{j=1}^r \sigma_j^2}{\sum_{j \geq 1} \sigma_j^2}.$$

Smaller E means we retain more snapshot energy and require more modes.

At Equal Dimension, the Local Model Is More Accurate



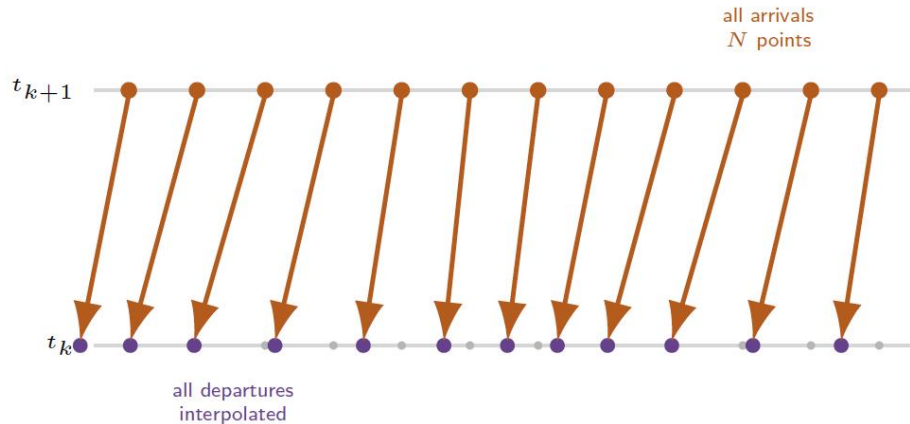
Error Definitions

- For a reduced approximation $u_k^{\text{ROM}} = U_k \hat{u}_k$,
ROM reconstruction error =
$$\frac{\|u_k^{\text{ROM}} - u_k^{\text{ROM}}\|_2}{\|u_k^{\text{FOM}}\|_2}.$$
- The best projection error is the best possible approximation given an orthonormal bases
$$= \min_z \frac{\|u_k - U_k z\|_2}{\|u_k\|_2} = \frac{\|(I - U_k U_k^\top) u_k\|_2}{\|u_k\|_2}.$$

Exact Operator vs. Sketched Operator

Exact semi-Lagrangian operator

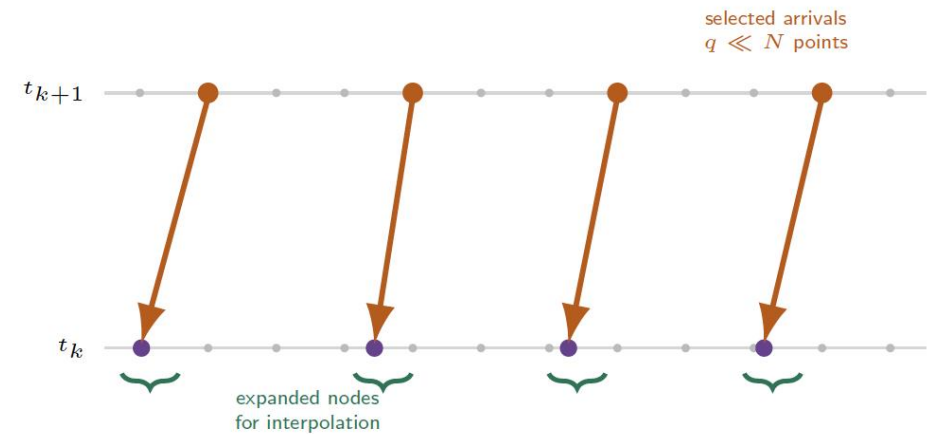
$$X_s = \{x_i\}_{i=1}^N: \text{ every arrival grid point}$$



full-grid action of $\mathcal{T}_k$: compute every characteristic

Sketched semi-Lagrangian operator

$$X_s = \{x_{i_j}\}_{j=1}^q: \text{ RRQR landmark arrivals}$$



sampld action: trace only RRQR-selected landmarks

Exact reduced update

$$\hat{u}_{k+1}^{\text{ex}} = U_{k+1}^\top \mathcal{T}_k(a) U_k \hat{u}_k^{\text{ex}}.$$

Sketched reduced update

$$\hat{u}_{k+1}^{\text{sk}} = (S_{k+1} U_{k+1})^\dagger S_{k+1} \mathcal{T}_k(a) U_k \hat{u}_k^{\text{sk}}.$$

Online transport reduction: characteristic backtraces $N \rightarrow q$
and basis interpolation work $Nr_k \rightarrow qr_k$, with $q \ll N$.

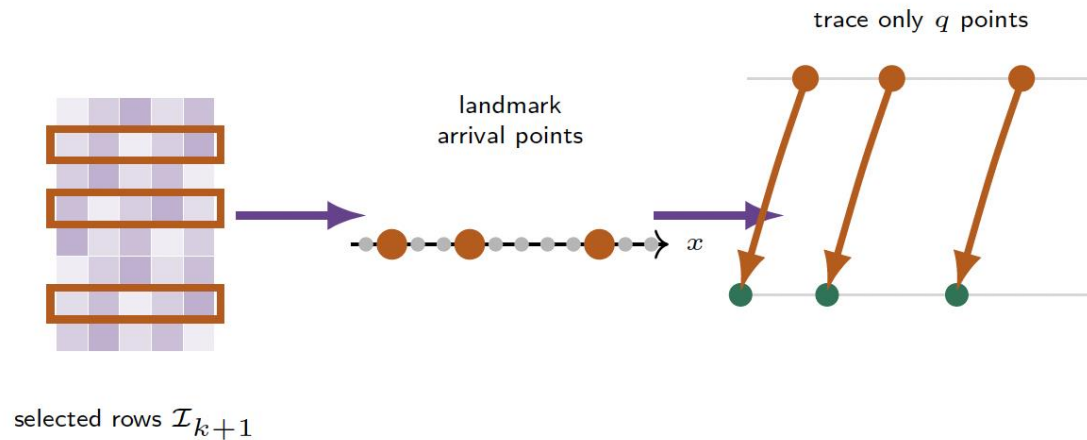
Second Reduction: Backtrace Only RRQR Landmarks

Offline selection

$$\tilde{U}_{k+1}^\top \Pi = QR, \quad \mathcal{I}_{k+1} = \{i_1, \dots, i_q\}.$$

Pivoted RRQR selects informative rows of U_{k+1}

U_{k+1} rows = grid points



Online sampled system

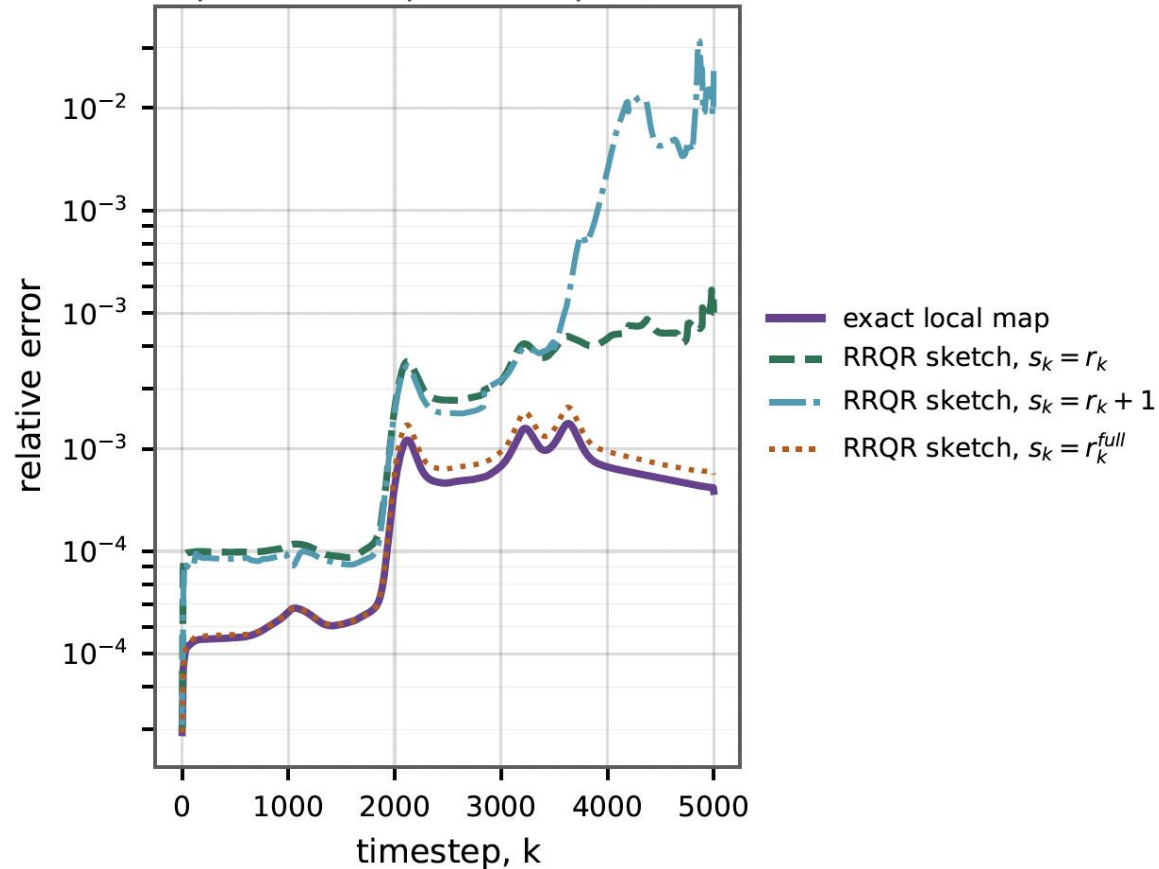
$$(S_{k+1}U_{k+1})\hat{u}_{k+1} \approx S_{k+1}\mathcal{T}_kU_k\hat{u}_k.$$
$$\hat{u}_{k+1}^{\text{sk}} = (S_{k+1}U_{k+1})^\dagger S_{k+1}\mathcal{T}_kU_k\hat{u}_k^{\text{sk}}.$$

N rows $\rightarrow$ q selected rows

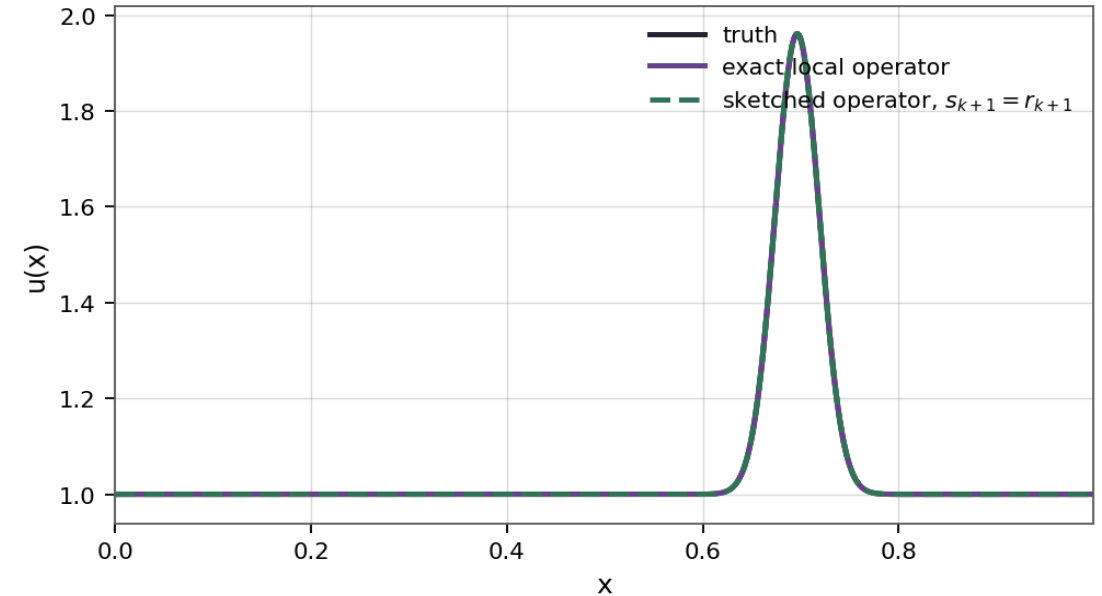
solve a small overdetermined problem

A Small Sketch Reproduces the Full Transport

Full transport vs. sampled transport, $E = 1e - 08$



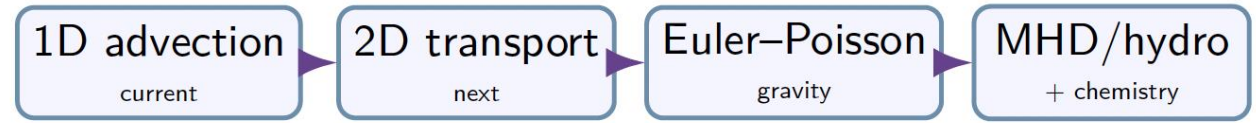
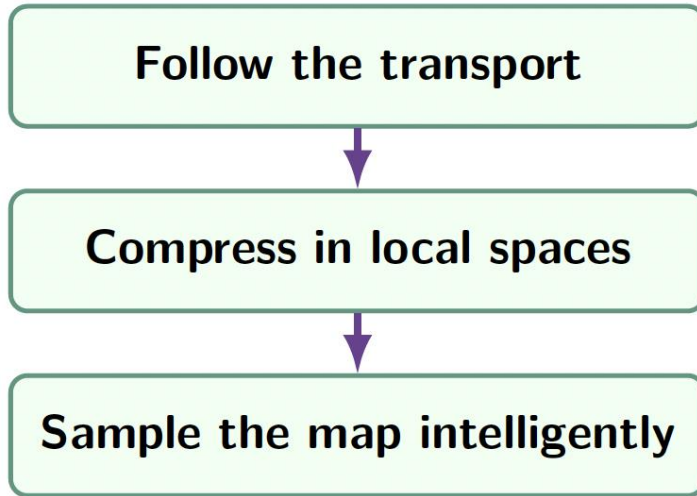
Exact vs sketched SLROM, $E=1e-08$
timestep = 0, sampled rows $s = 1$



N backtraces $\rightarrow q$ backtraces

$$q/N = 10\%$$

Summary and Future Works



First POD mode of this talk: follow the flow, then compress what remains.

THANK YOU!

