A Crash Course in Radio Astronomy and Interferometry: 2. Aperture Synthesis

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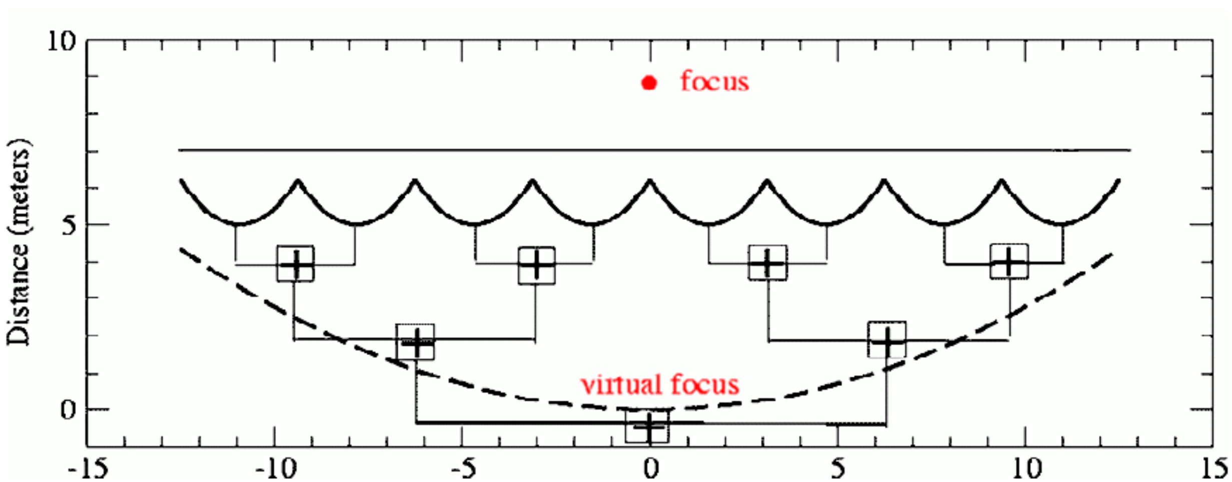
(thanks to S. Dougherty, C. Chandler, D. Wilner & C. Brogan)

Output of a Filled Aperture

- Signals at each point in the aperture are brought together in phase at the antenna output (the focus)
- Imagine the aperture to be subdivided into *N* smaller elementary areas; the voltage, $V(t)$, at the output is the sum of the contributions $\Delta V_i(t)$ from the *N* individual aperture elements:

$$V(t) = \sum \Delta V_i(t)$$



Aperture Synthesis: Basic Concept

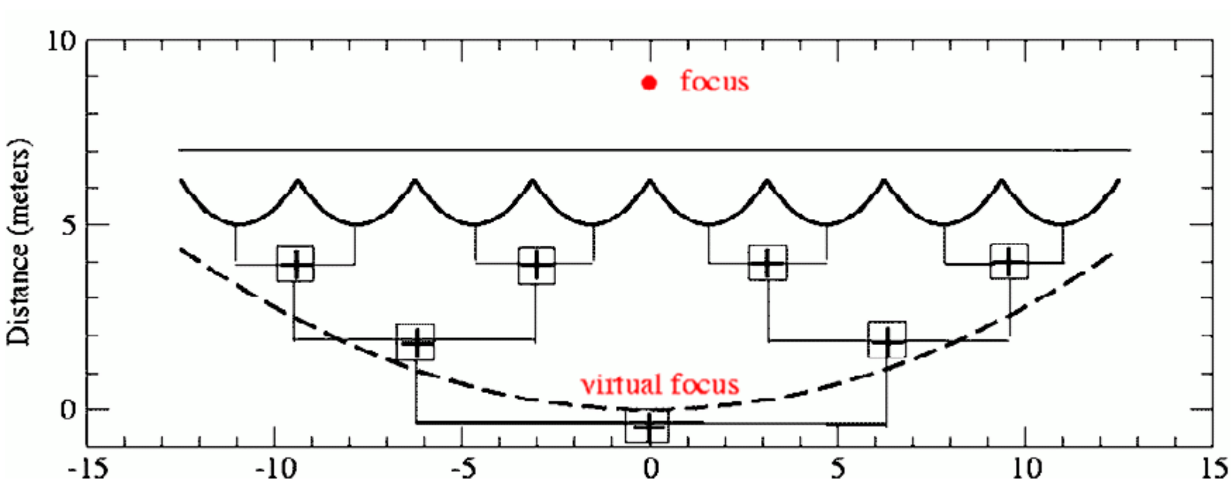
- The radio power measured by a receiver attached to the telescope is proportional to a running time average of the square of the output voltage:

$$\langle P \rangle \propto \left\langle \left(\sum \Delta V_i \right)^2 \right\rangle = \sum \sum \left\langle \left(\Delta V_i \Delta V_k \right) \right\rangle$$
$$= \sum \left\langle \Delta V_i^2 \right\rangle + \sum_{i \neq k} \sum \left\langle \Delta V_i \Delta V_k \right\rangle$$

- Any measurement with the large filled-aperture telescope can be written as a sum, in which each term depends on contributions from only two of the *N* aperture elements
- Each term $\langle \Delta V_i \Delta V_k \rangle$ can be measured with two small antennas, if we place them at locations *i* and *k* and measure the average product of their output voltages with a correlation (multiplying) receiver

Aperture Synthesis: Basic Concept

- If the source emission is unchanging, there is no need to measure all the pairs at one time
- One could imagine sequentially combining pairs of signals. For N sub-apertures there will be $N(N-1)/2$ pairs to combine
- Adding together all the terms effectively "synthesizes" one measurement taken with a large filled-aperture telescope
- Can synthesize apertures much larger than can be constructed as a filled aperture, giving very good spatial resolution



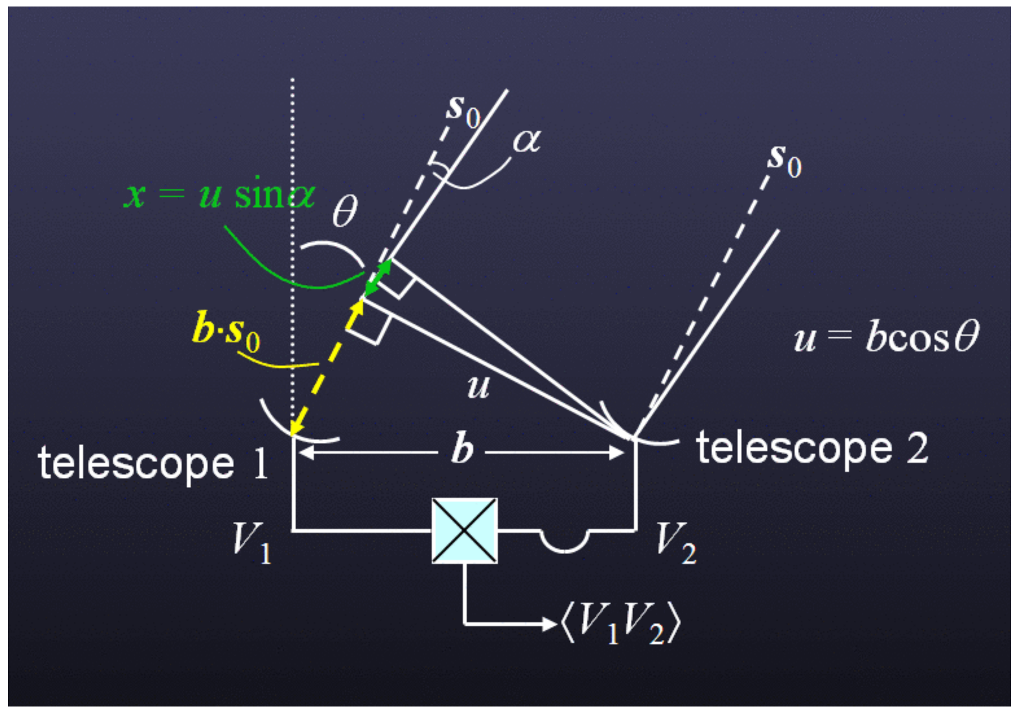
A Simple 2-Element Interferometer

What is the interferometer response as a function of sky position, $l = \sin\alpha$?

In direction $\boldsymbol{s}_0$ ($\alpha = 0$) the wavefront arriving at telescope #1 has an extra path $\boldsymbol{b} \times \boldsymbol{s}_0 = b \sin\theta$ to travel relative to #2

The time taken to traverse this extra path is the *geometric delay*, $\tau_g = \boldsymbol{b} \times \boldsymbol{s}_0/c$

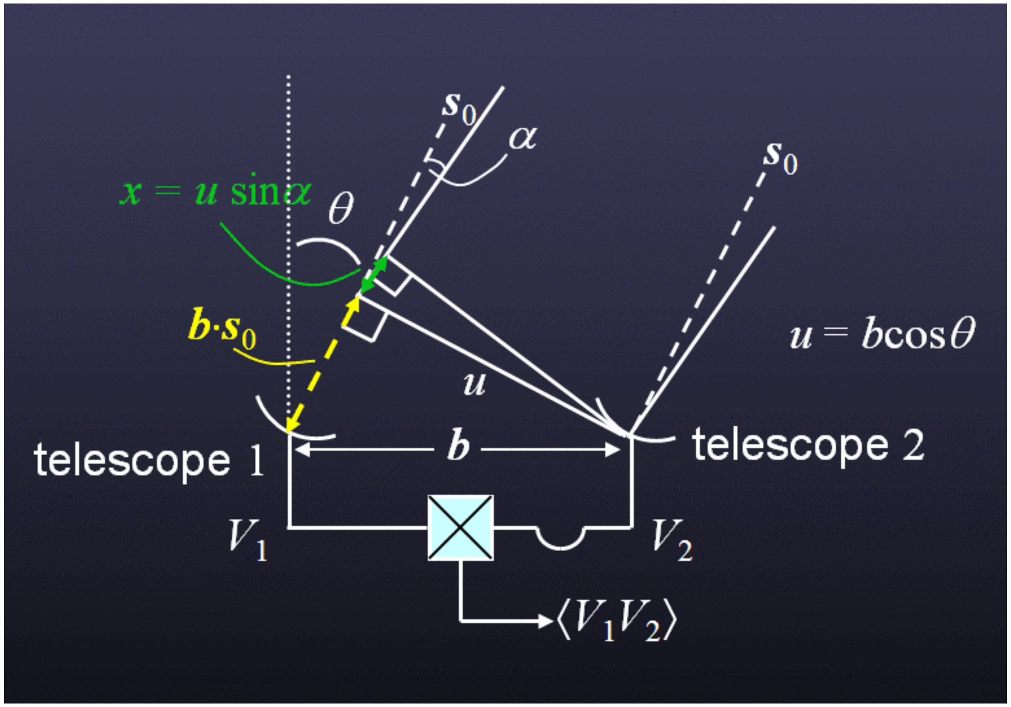
This delay is compensated for by inserting a signal path delay for #2 equivalent to τ_g



Response of a 2-Element Interferometer

At angle α, a wavefront has an extra path $x = u \sin\alpha = ul$ to travel

Expand to 2D by introducing β orthogonal to α, $m = \sin\beta$, and v orthogonal to u, so that in this direction the extra path $y = vm$



Write all distances in units of wavelength, $x \equiv x/\lambda$, $y \equiv y/\lambda$, etc., so that x and y are now numbers of cycles

Extra path is now $ul + vm$

$\Rightarrow V_2 = V_1 \, e^{-2\pi i(ul+vm)}$

Correlator Output

- The output from the correlator (the multiplying and time-averaging device) is:

$$C = \langle V_1 V_2 \rangle = \left\langle \iint V_1(l,m)\,dldm \iint V_2(l,m)\,dldm \right\rangle$$

- For ($l_1 \neq l_2$, $m_1 \neq m_2$) the above average is zero (assuming mutual sky), so

$$C = \left\langle \iint V_1(l,m) V_2(l,m)\,dldm \right\rangle$$

$$= \iint \langle V_1(l,m) V_2(l,m) \rangle\,dldm$$

$$= \iint \langle V_1(l,m)^2 \rangle e^{-2\pi i(ul+vm)}\,dldm$$

$$= \iint I(l,m) e^{-2\pi i(ul+vm)}\,dldm$$

The Complex Visibility

- Thus, the interferometer measures the *complex visibility*, *V*, of a source, which is the FT of its intensity distribution on the sky:

$$\mathcal{V}(u,v) = Ae^{-i\phi} = \iint I(l,m)e^{-2\pi i(ul+vm)}dldm$$

  - *u,v* are *spatial frequencies* in the E-W and N-S directions, are the projected baseline lengths measured in units of wavelength, i.e., B/λ
  - *l, m* are direction cosines relative to a reference position in the E-W and N-S directions
  - (l = 0, m = 0) is known as the *phase center*
  - the phase ϕ contains information about the location of structure with spatial frequency *u,v* relative to the phase center

The Complex Visibility

- This FT relationship is the ***van Cittert-Zernike theorem***, upon which synthesis imaging is based
- It means there is an inverse FT relationship that enables us to recover *I*(*l*,*m*) from *V*(*u*,*v*):

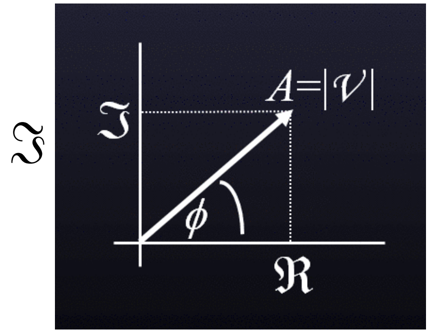
$$\mathcal{V}(u,v) = \iint I(l,m)e^{-2\pi i(ul+vm)}dldm$$

$$I(l,m) = \iint \mathcal{V}(u,v)e^{2\pi i(ul+vm)}dudv$$

- The correlator measures both real and imaginary parts of the visibility to give the amplitude and phase:

$$A = \sqrt{\Re^2 + \Im^2}$$

$$\phi = \tan^{-1}\left(\frac{\Im}{\Re}\right)$$



The Primary Beam

- The elements of an interferometer have finite size, and so have their own response to the radiation from the sky
- This results in an additional factor, $\mathcal{A}(l,m)$, to be included in the expression for the correlator output, which is the *primary beam* or normalized reception pattern of the individual elements

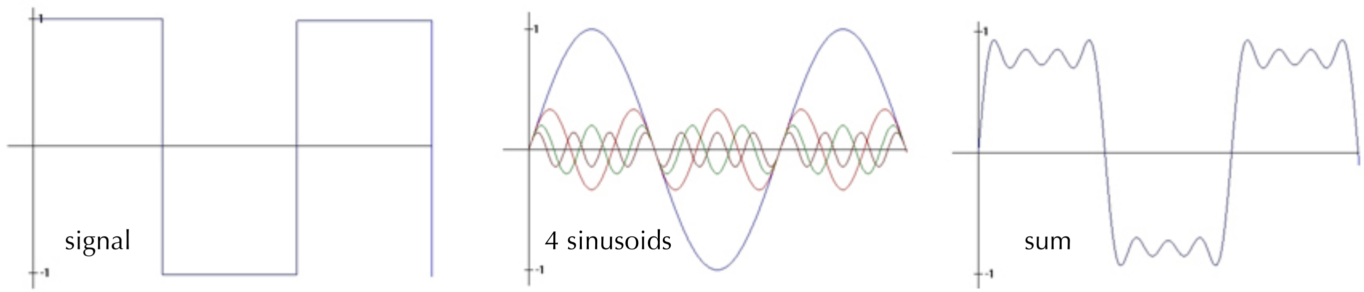
$$C = \iint \mathcal{A}(l,m) I(l,m) e^{-2\pi i(ul+vm)} dl dm$$

- Interferometer actually measures the FT of the sky multiplied by the primary beam response
- Need to divide by $\mathcal{A}(l,m)$ to recover $I(l,m)$, the last step of image production
- Primary Beam FWHM is the “Field-of-View” of a single-pointing interferometric image

The Fourier Transform

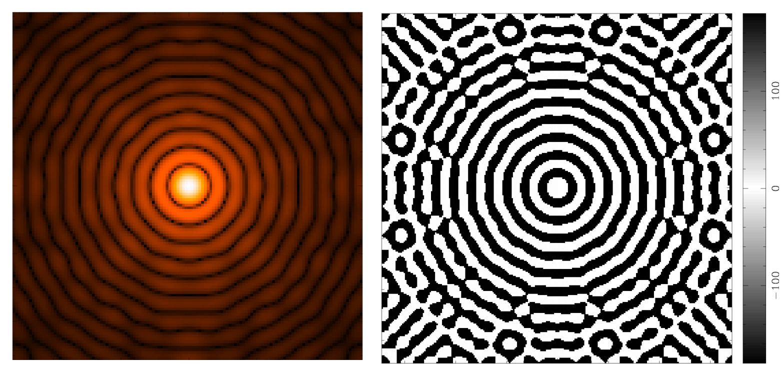
- Fourier theory states that any signal (including images) can be expressed as a sum of sinusoids

Jean Baptiste Joseph Fourier 1768-1830



- (x,y) plane and (u,v) plane are conjugate coordinate systems

I(x,y) V(u,v) = FT{I(x,y)}

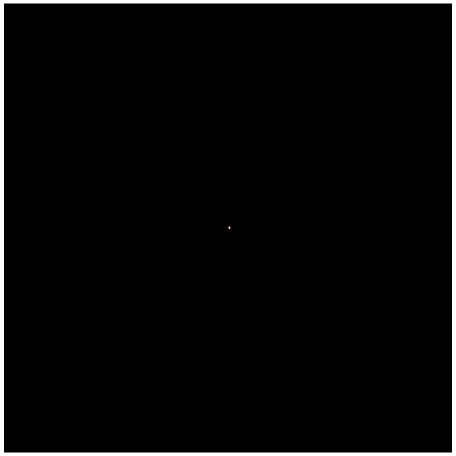


- the Fourier Transform contains all information of the original

Some 2-D Fourier Transform Pairs

***I*(x,y)** | **Amp{*V*(u,v)}**

δ Function



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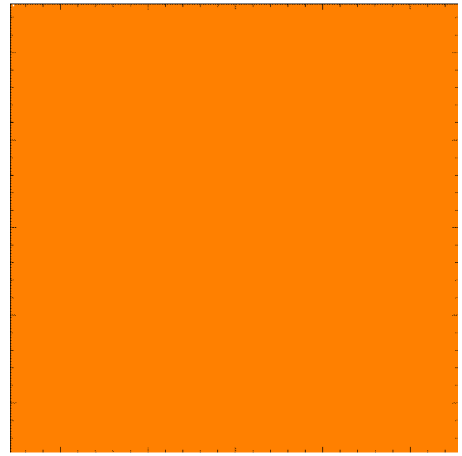
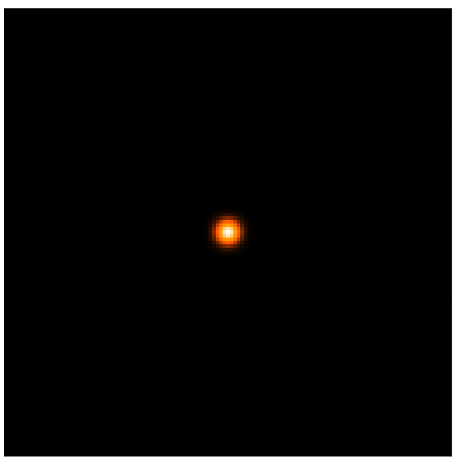


Constant

Gaussian



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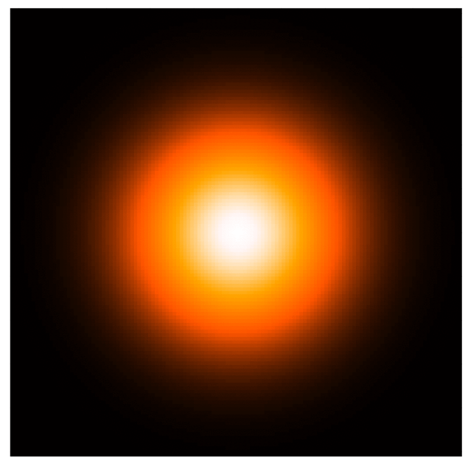


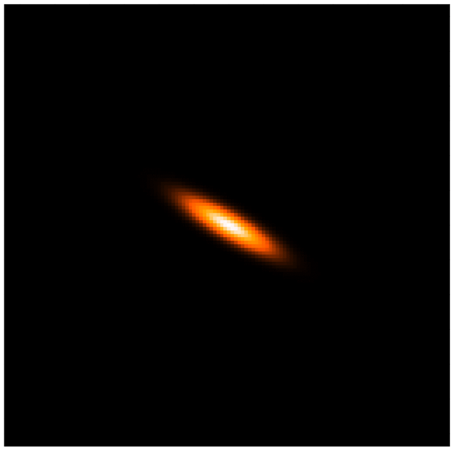
Gaussian

narrow features transform to wide features (and vice-versa)

More 2-D Fourier Transform Pairs

***I*(x,y)** | **Amp{*V*(u,v)}**

elliptical Gaussian



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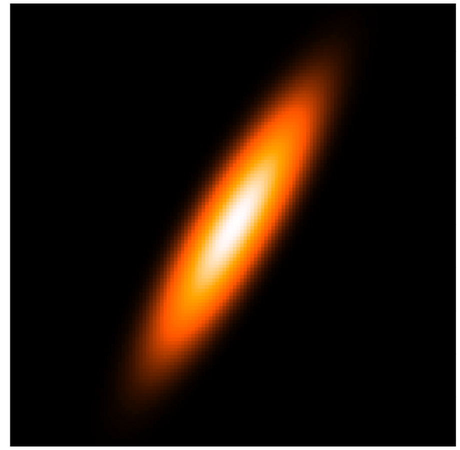

elliptical Gaussian

Disk

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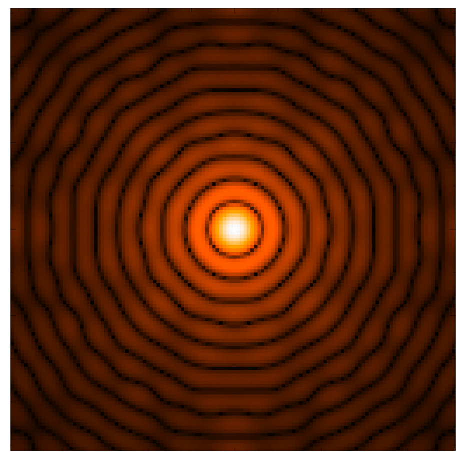


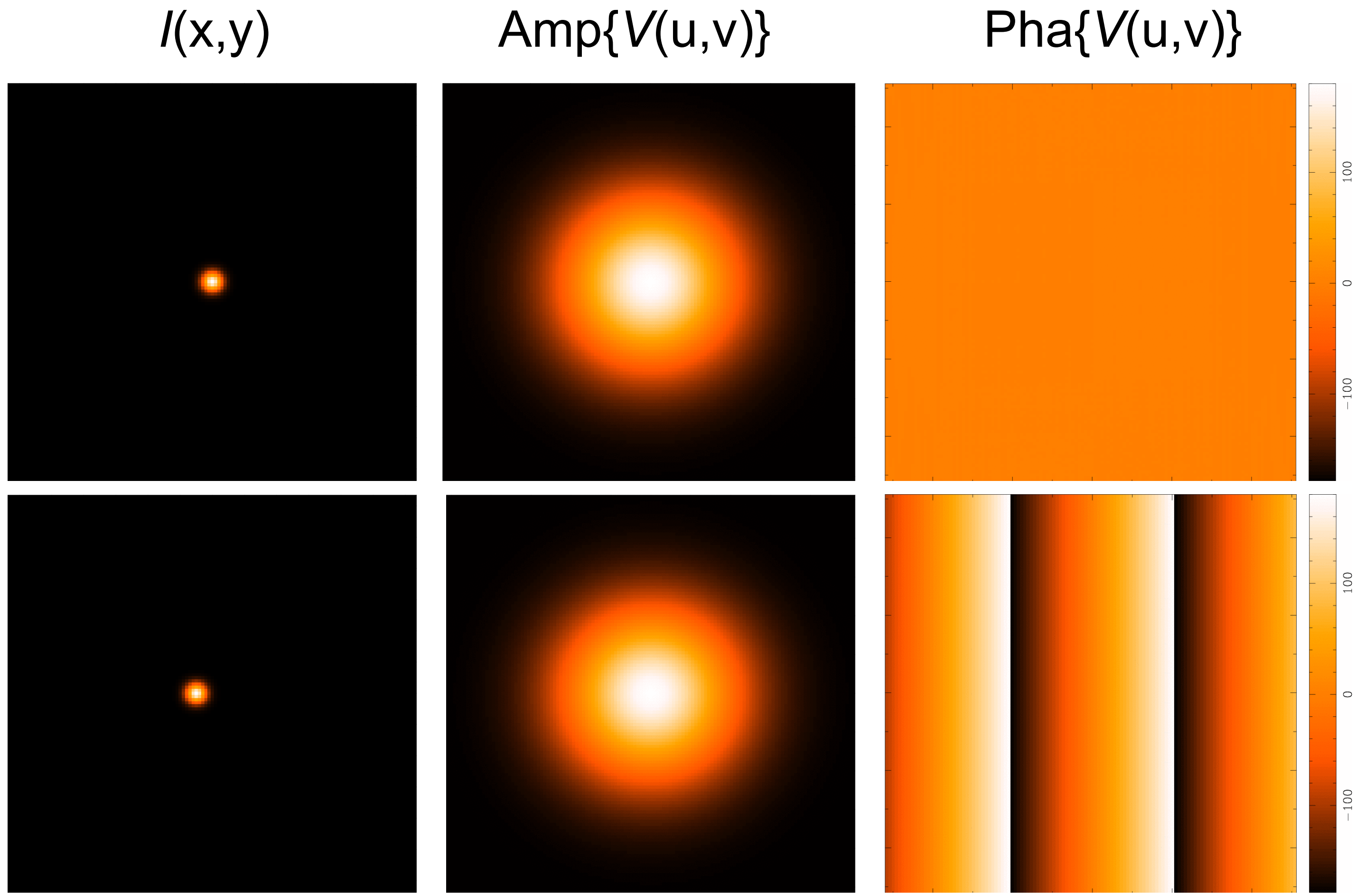
Bessel

sharp edges result in many high spatial frequencies

Amplitude and Phase

complex numbers: (real, imaginary) or (amplitude, phase)

- amplitude tells "how much" of a certain spatial frequency component
- phase tells "where" this component is located



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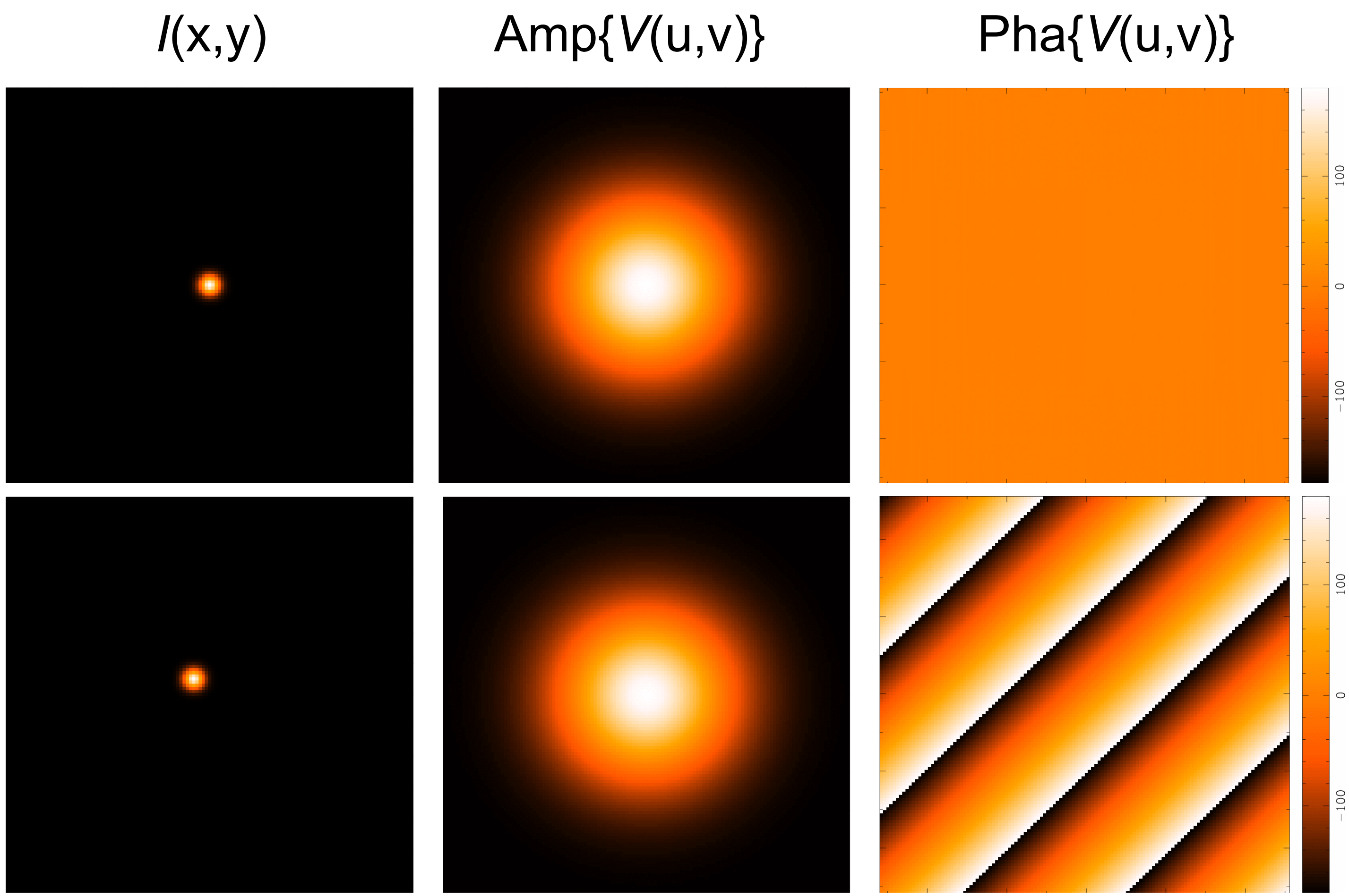
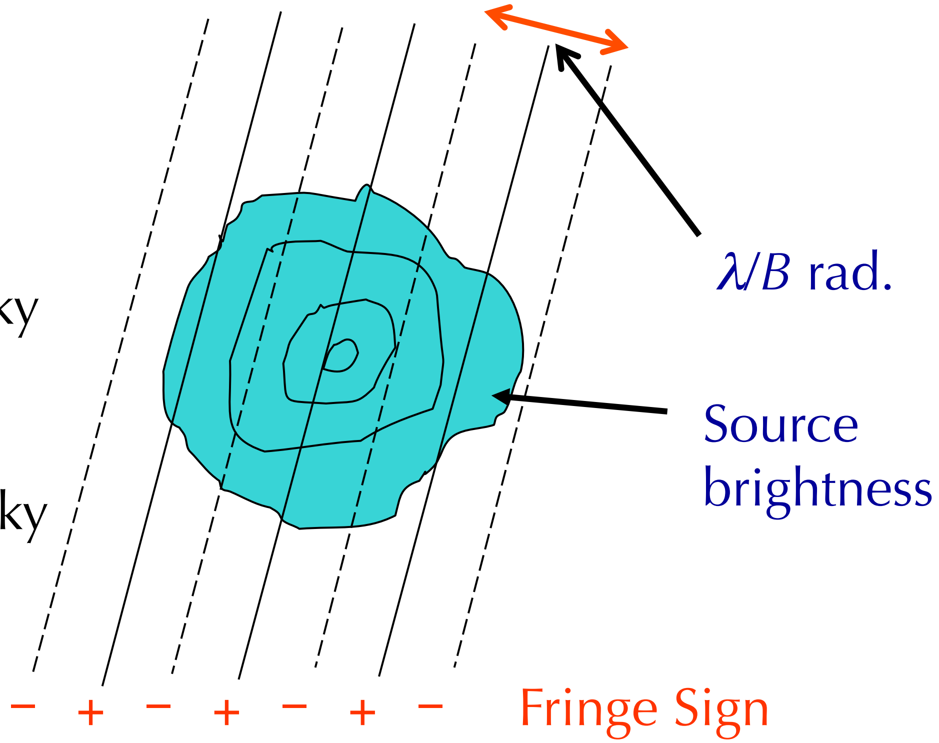


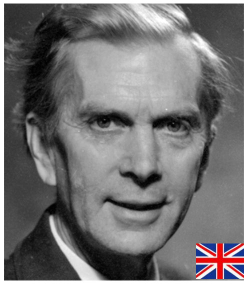
Picturing the Visibility: Fringes

- The FT of a single visibility measurement is a sinusoid with spacing $1/u = \lambda/B$ between successive peaks, or "fringes"
- Build up an image of the sky by summing many such sinusoids (addition theorem)
- FT scaling theorem shows:
  - Short baselines have large fringe spacings and measure large-scale structure on the sky
  - Long baselines have small fringe spacings and measure small-scale structure on the sky

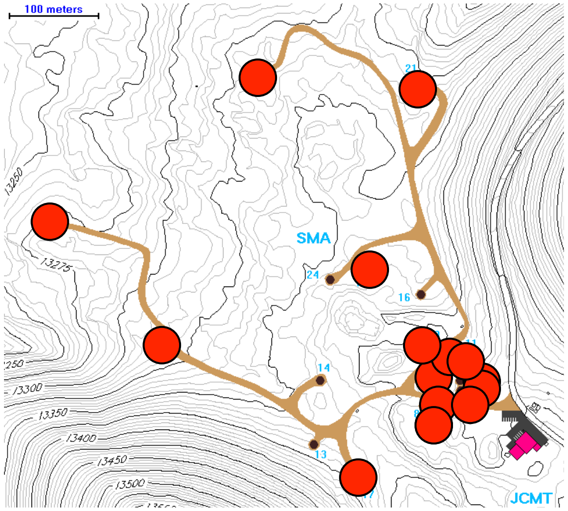


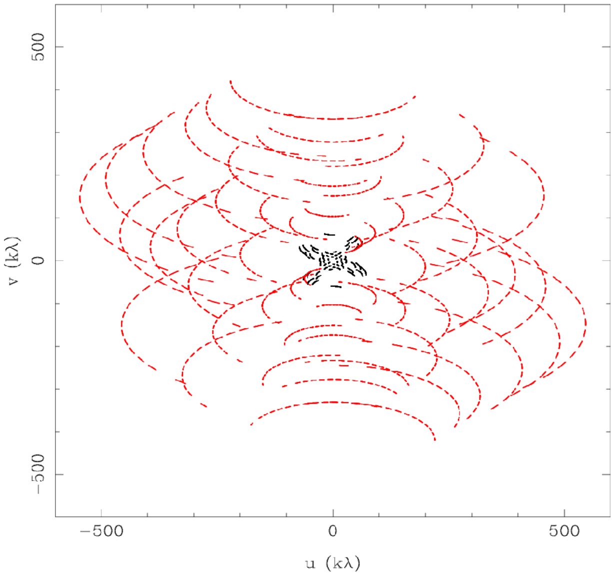
Aperture Synthesis

- sample *V*(*u*,*v*) at enough points to synthesize the equivalent large aperture of size (u_{max},v_{max})
  - 1 pair of telescopes → 1 (*u*,*v*) sample at a time
  - *N* telescopes → number of samples = *N*(*N*-1)/2
  - fill in (*u*,*v*) plane by making use of Earth rotation: Martin Ryle, 1974 Nobel Prize in Physics
  - reconfigure physical layout of *N* telescopes for more



Sir Martin Ryle
1918-1984





2 configurations
of 8 SMA antennas
345 GHz
Dec = -24 deg


Next: Interferometric
Imaging